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THE MULTIPLE SCATTERING OF WAVES IN IRREGULAR MEDIA

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CONTENTS

When a wave passes through a large thickness of a non-absorbing medium containing weak random irregularities of refractive index, large amplitude and phase fluctuations of the wave field can develop. The probability distributions of these fluctuations are important, since they may be readily observed and from them can be found the mean square amplitudes of the fluctuations. This paper shows how to calculate these distributions and also the 'angular power spectrum' for an assembly of media which are statistically stationary with respect to variations in time, and in space for directions perpendicular to the wave normal of the incident wave. The scattered field at a given point is resolved into two components in phase and in quadrature with the residual unscattered wave at that point. The assembly averages of the powers in these two components, and of their correlation coefficient are found, and a set of three integro-differential equations is constructed which show how these three quantities vary as the medium is traversed. The probability distributions of amplitude and phase of the wave field at any point in the medium are functions of these three quantities which are found by integrating the equations through the medium. An essential feature of these equations is that they include waves which have been scattered several or many times (multiple scatter). The equations are solved analytically for some particular cases. Solutions for the general case have been obtained numerically and are presented, together with the corresponding probability distributions of the field fluctuations and their average values.

1. Introduction

The passage of radio waves through the interplanetary medium (Hewish, Scott & Wills 1964; Cohen 1965) is an example of a general class of problems, often encountered in other branches of physics, in which a plane wave passes through a large thickness of a non-absorbing medium which has small irregular fluctuations of refractive index. The present paper deals with the theory of this subject when

(*a*) the irregularities are large compared with the wavelength of the radiation so that the wave is scattered through only a small angle by individual irregularities,

(*b*) the irregularities are weak so that, in a layer whose thickness is several times greater than the average size of an irregularity, only single scattering is appreciable, and

(*c*) the medium is very thick so that the wave field may consist predominantly of waves which have been scattered several times.

These conditions hold for radio waves in the interplanetary medium and in some other cases, for example, sound waves in the atmosphere or the oceans. Media of this type have been called 'weakly irregular' media (Uscinski 1967). Condition (*a*) means that effects of polarization may be neglected. It should be noted that the theory may be extended to cover the case where the total angular deviation of a multiply scattered wave is not small.

The main purpose of this paper is to show how to take into account the effects of multiple scatter arising from condition (*c*), and the probability distributions of amplitude and phase of the wave field are derived for this case. Most earlier treatments of multiple scatter have considered only the total scattered power. Because the medium is thick, the emerging wave front may show large fluctuations of amplitude and phase, even though the medium is weakly scattering. This paper does not deal with strong irregularities which could produce large fluctuations after the wave has passed through only a thin layer of the medium.

Much work has already been done on 'weakly scattering' media. The angular distribution of the scattered power has been studied by Booker & Gordon (1950) and Bowhill (1961 *b*) for the case when the incident wave is only slightly attenuated so that single scatter alone need be taken into account. Some further statistical properties of the field emerging from a thin scattering layer have been discussed by Bowhill (1957, 1961 *a*). The mean square values of the amplitude and phase fluctuations of the wave field for the single scatter case have been treated by Obukhov (1953), Chernov (1960) and Tatarski (1961) using the 'method of smooth perturbations', while Chernov also derives the autocorrelation functions for these fluctuations. The probability distributions of phase and amplitude of the wave field are given by Uscinski (1967) who discussed some limitations of the 'method of smooth perturbations'.

The angular distribution of the scattered power for the case when multiple scatter must be taken into account has been discussed by Fejer (1953) and Howells (1960). Basic mathematical treatments of multiple scatter have been given by many authors including Foldy (1945), Tatarski (1964), Tatarski & Gertsenshtein (1963), Furutsu (1963), but much of this work is of a formal nature and is difficult to evaluate in concrete cases. The 'method of smooth perturbations' is modified by Tatarski (1965, 1967) to take multiple scatter into account, and he gives expressions for the mean square logarithm of the amplitude of the wave field.

This paper considers an assembly of scattering media which are statistically stationary with respect to variations in time, and in space for directions perpendicular to the wave normal of the incident wave. Averages taken over the assembly are indicated by angular brackets $\langle\ \rangle$, and the word 'average' or 'mean' always refers to an assembly average. In § 2 the problem of determining, for this assembly, the amplitude and phase probability distributions of the field is reduced to finding the quantities S_1, S_2 (the average scattered power co-phased and in quadrature, respectively, with the unscattered wave) and K (the un-normalized correlation coefficient of the scattered power). These quantities are defined in § 2, and in the later sections they are found, as follows.

A system of polar coördinates is introduced with polar angles θ, ϕ, and with the z axis in the direction of the incident wave normal. The medium is divided up into elementary layers in such a way that the z axis is normal to each layer. An elementary layer is sufficiently thin so that only single scattering of the waves incident on it need be considered. In § 3 the scattering cross-section is derived for the power scattered into the direction θ, ϕ by an elementary layer when a wave is incident from an angle θ', ϕ'. The specific form of this cross-section is then found when the irregularities have a Gaussian autocorrelation function, and in § 4 it is further simplified for the case when the total angle through which the radiation is scattered is small.

The counterparts of S_1, S_2 and K are now considered for the power flux in a small solid angle in the neighbourhood of the direction θ, ϕ, and are designated by σ_1, σ_2 and κ. In § 5 relations are found which describe how σ_1, σ_2 and κ change when the waves are scattered by an elementary layer. In § 6 these relations, together with the scattering cross-section, are used to construct a set of three integro-differential equations which take multiple scatter into account and describe the variation of σ_1, σ_2 and κ as the medium is traversed. In § 7 the physical significance of these equations is discussed. In § 8 they are solved analytically, for some special cases, to give σ_1, σ_2 and κ for waves which have traversed a thickness z of the medium, and finally these quantities are integrated with respect to θ and ϕ to give S_1, S_2 and K, respectively. Solutions for the general case have been obtained by stepwise numerical integration of the integro-differential equations. The results are presented in §§ 10 and 11 together with the corresponding probability distributions of the phase and amplitude fluctuations, and their average values at different distances in the medium.

2. Statistical description of the wave field

The wave field in the scattering medium is described by the electric intensity E. The statistical treatment used here of this field is similar to that used by Beckmann & Spizzichino (1963). It has been shown elsewhere (Uscinski 1967) that polarization effects may be neglected and E may be treated as a complex scalar. It can then be represented in amplitude and phase by a vector in the Argand diagram in which the Cartesian coordinates are u and v (figure 1). The vector E is composed of the field E_0 of the unscattered incident wave, and E_1 which is the resultant of the fluctuating scattered fields. The coordinate axis u is chosen to be parallel to E_0 and the phase of the total field vector E, taken with respect to the unscattered wave E_0 is ψ.

Let R, I be the Cartesian components of E_1, and define the following average quantities

$$\langle R^2\rangle = S_1, \quad \langle I^2\rangle = S_2, \quad \langle RI\rangle = K. \tag{2·1}$$

These three quantities will be called the mean co-phased scattered power, the mean quadrature scattered power and the un-normalized correlation coefficient of the scattered power respectively.

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The quantities (2·1) are taken relative to the unprimed axes. If a second set of axes with primed coordinates u', v' is taken as a new phase reference differing from the former by an angle χ (figure 1), then the new values of (2·1), given by S_1', S_2' and K', are related to the old by the transformation

$$\left.\begin{aligned} S_1' &= S_1\cos^2\chi + S_2\sin^2\chi + 2K\sin\chi\cos\chi,\\ S_2' &= S_1\sin^2\chi + S_2\cos^2\chi - 2K\sin\chi\cos\chi,\\ K' &= (S_2 - S_1)\sin\chi\cos\chi + K(\cos^2\chi - \sin^2\chi).\end{aligned}\right\} \quad (2\cdot2)$$

Special cases of these equations have been given by Hristow (1961) and Gnedenko & Kolmogorov (1949).

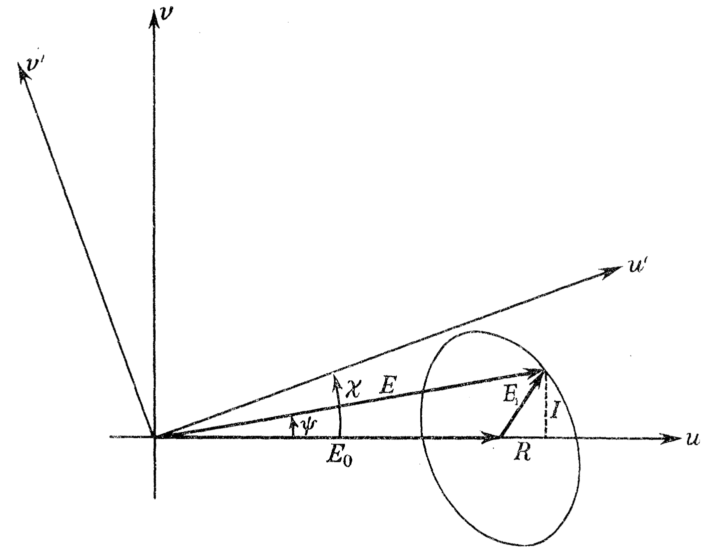

FIGURE 1. Scattered and unscattered field components as complex scalars in the Argand plane together with the rotation of axes used in the transformations (2·2).

The two-dimensional probability distribution of the Cartesian components u, v of E has been given, for this case, by Uscinski (1967) following Beckmann & Spizzichino (1963):

$$w(u,v) = \frac{1}{2\pi(S_1S_2-K^2)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2(1-K^2/S_1S_2)}\left[\frac{(u-E_0)^2}{S_1} - \frac{2K(u-E_0)\,v}{S_1S_2} + \frac{v^2}{S_2}\right]\right\}. \quad (2\cdot3)$$

The loci of constant w are then ellipses in the u–v plane centred on the point $(E_0, 0)$. A typical ellipse is shown in figure 1.

The probability distribution of the amplitude $|E|$ is obtained by expressing u and v in polar coordinates

$$u = |E|\cos\psi, \quad v = |E|\sin\psi \quad (2\cdot4)$$

and integrating (2·3) over all phases ψ, while the phase distribution is obtained by integrating over all amplitudes $|E|$.

When the symbols S_1, S_2 and K are used, the above treatment refers to the total scattered field arriving at a point in the medium. It may, however, be applied to that part of the field consisting of plane waves with their wave normals in a small solid angle $d\Omega$ between θ, $\theta+d\theta$ and ϕ, $\phi+d\phi$, where θ, ϕ are polar angles which use as polar axis the direction z of the wave normal of the unscattered incident wave (figure 2). In this case the quantities corresponding to (2·1) for a point at distance z in the medium will be written

$$\sigma_1(z,\theta,\phi)\,\mathrm{d}\Omega,\quad \sigma_2(z,\theta,\phi)\,\mathrm{d}\Omega,\quad \kappa(z,\theta,\phi)\,\mathrm{d}\Omega \tag{2·5}$$

respectively, where $$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{2·6}$$

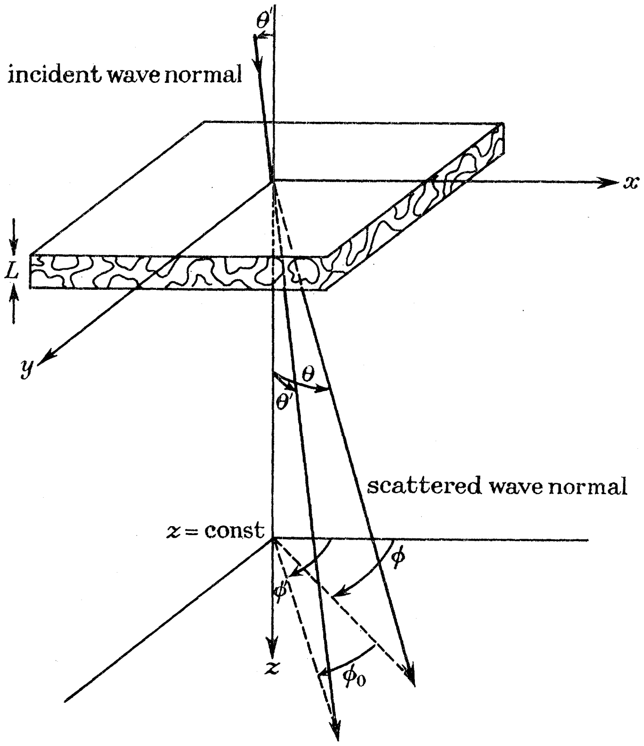

FIGURE 2. Geometry of the incident and scattered waves.

3. POWER SCATTERED BY A THIN LAYER OF THE MEDIUM

Let x, y, z be Cartesian coordinates in space. The scattering medium is assumed to extend to infinity in the x and y directions and to lie on the positive side of the plane $z = 0$. The refractive index n of the medium is assumed to depart from unity by very small amounts. Within the scattering medium let

$$n = n_0 + n_1(x_0, y_0, z_0) \tag{3·1}$$

where n_0, the average value of n, is taken to be unity. The small deviation n_1 is assumed to be real, and x_0, y_0, z_0 are the coordinates of a point in the scattering medium.

Now let the medium be divided into layers parallel to the plane $z = 0$, and of thickness L, large compared with the average size of an irregularity, yet small enough to ensure that only single scattering is important for waves incident on such an 'elementary layer'. Let a plane wave, with electric field given by (figure 2)

$$E_i = E_0 \exp\{-\mathrm{i}k[z\cos\theta' + x\sin\theta'\cos\phi' + y\sin\theta'\sin\phi']\}, \tag{3·2}$$

be obliquely incident on a layer. The field of the wave scattered by this layer will now be found. The result for normal incidence was derived in a previous paper (Uscinski 1967) and the method may easily be extended as follows.

Let the scattered wave be observed in the plane $z = $ const. which lies beyond the scattering layer. The incident wave normal meets this plane at the point $z \tan\theta' \cos\phi'$, $z\tan\theta'\sin\phi'$, z, and the small horizontal displacements of the general point x, y from this point are given by

$$X = x - z\tan\theta'\cos\phi', \quad Y = y - z\tan\theta'\sin\phi'. \tag{3·3}$$

Similarly let

$$X_0 = x_0 - z_0\tan\theta'\cos\phi', \quad Y_0 = y_0 - z_0\tan\theta'\sin\phi' \tag{3·4}$$

which are small horizontal displacements of a point in the scattering layer from the incident wave normal.

Then the incident wave (3·2) is

$$E_i = E_0 \exp\{-\mathrm{i}k[z\sec\theta' + X\sin\theta'\cos\phi' + Y\sin\theta'\sin\phi']\}. \tag{3·5}$$

Now let the x- and y-components of the incident wave vector be

$$k'_x = k\sin\theta'\cos\phi', \quad k'_y = k\sin\theta'\sin\phi'. \tag{3·6}$$

Then (3·5) becomes

$$E_i = E_0\exp\{-\mathrm{i}(kz\sec\theta' + k'_x X + k'_y Y)\}. \tag{3·7}$$

The scattered field from one member of the assembly of scattering layers is now found by the method set out in detail by Booker & Gordon (1950) and Budden (1965*a*). The field re-radiated from the dipole moment induced in a small volume $\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0$ of the medium is given by Budden (*ibid.* equation (11)). This is now integrated through the scattering layer to give the total scattered field at the point x, y, z:

$$E_s(X,Y) = \frac{k^2E_i}{2\pi}\int_0^L\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{n_1(x_0,y_0,z_0)}{(z-z_0)\sec\theta'}\exp\{-\mathrm{i}kH\}\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0, \tag{3·8}$$

where

$$H = \frac{(X-X_0)^2 + (Y-Y_0)^2 - \sin^2\theta'\{(X-X_0)\cos\phi' + (Y-Y_0)\sin\phi'\}^2}{2(z-z_0)\sec\theta'}. \tag{3·9}$$

The field (3·8) is now expressed as an 'angular spectrum' of plane waves by taking its two-dimensional Fourier transform. It is shown in appendix A that the coefficient of the plane wave Fourier component, whose wave normal vector has Cartesian components k_x, k_y, is

$$F(k_x,k_y) = \frac{E_0k^2\exp(-\mathrm{i}kz\sec\theta')}{2\pi W}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^L\int_{-\frac12W}^{\frac12W}\int_{-\frac12W}^{\frac12W}\frac{n_1(x_0,y_0,z_0)}{(z-z_0)\sec\theta'}\exp\{-\mathrm{i}kH\}$$
$$\times\exp\{\mathrm{i}([k_x-k'_x]X + [k_y-k'_y]Y)\}\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0\,\mathrm{d}X\,\mathrm{d}Y, \tag{3·10}$$

where the range of integration W may be made indefinitely large. The limits of the integrals are discussed in appendix A. When the X and Y integrations are carried out the product FF^* is, from (A 8) of appendix A,

$$FF^* = \frac{E_0^2\sec^2\theta' k^2}{W^2}\int(6)\int n_1n'_1\exp\{\mathrm{i}[(k_x-k'_x)(x_0-x'_0) + (k_y-k'_y)(y_0-y'_0)$$
$$-(\alpha+\gamma)(z_0-z'_0)]\}\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0\,\mathrm{d}x'_0\,\mathrm{d}y'_0\,\mathrm{d}z'_0, \tag{3·11}$$

where

$$\alpha+\gamma = \tfrac12 k\{\sec\theta'[\sin^2\theta - \sin^2\theta'] + \tan^2\theta'[\sin\theta\cos\phi_0 - \sin\theta']^2\}. \tag{3·12}$$

Here $\phi_0 = \phi - \phi'$, $n'_1 = n_1(x'_0, y'_0, z'_0)$, and x'_0, y'_0, z'_0 are simply a second set of values of x_0, y_0, z_0. The limits of the integrals are $\pm\frac12 W$ for x_0, y_0, x'_0, y'_0, and 0 to L for z_0, z'_0.

For the component plane waves having k_x, k_y in the small range δk_x, δk_y, the power crossing unit area perpendicular to the z axis is, from equation (A 7)

$$\frac{FF^*}{8\pi^2 Z_0}\,\delta k_x\,\delta k_y. \tag{3·13}$$

Equations (3·11) and (3·13) give the power spectrum for a single scattering layer. The average value of (3·11) will now be found for an assembly of a large number of layers. It is assumed that in (3·1)

$$n_1 = \mu(z_0)\,n_b(x_0, y_0, z_0), \tag{3·14}$$

where the profile function $\mu(z_0)$ is the same for all members of the assembly and gives the magnitude of the refractive index fluctuations, while n_b is a stochastic function which is statistically homogeneous and stationary with respect to the three variables x_0, y_0, z_0 at least over the thickness of an elementary layer. Its mean value is zero and its variance is unity.

Let ρ be the three-dimensional autocorrelation function of n_b, defined by

$$\rho(\xi,\eta,\zeta) = \langle n_b(x_0, y_0, z_0)\,n_b(x_0+\xi, y_0+\eta, z_0+\zeta)\rangle, \tag{3·15}$$

where

$$\xi = x_0'-x_0, \quad \eta = y_0'-y_0, \quad \zeta = z_0'-z_0. \tag{3·16}$$

It is assumed that ρ is a Gaussian function and that the irregularities are isotropic in the $x-y$ plane with scale size r_0. They may, however, be elongated in the z direction since it is instructive to study how the axial ratio r_z/r_0 affects the scattering properties of the medium. Thus

$$\rho(\xi,\eta,\zeta) = \exp\{-(\xi^2+\eta^2)/r_0^2\}\exp\{-\zeta^2/r_z^2\}. \tag{3·17}$$

Further let

$$2X_1 = x_0'+x_0, \quad 2Y_1 = y_0'+y_0, \quad 2Z_1 = z_0'+z_0. \tag{3·18}$$

Now set (3·16), (3·17) and (3·18) in (3·11) and take the assembly average. Then

$$\langle FF^*\rangle = \frac{E_0^2 k^2 \sec^2\theta' \mu^2}{W^2}\int(6)\int \rho(\xi,\eta,\zeta)\exp\{-\mathrm{i}[(k_x-k_x')\,\xi + (k_y-k_y')\,\eta-(\alpha+\gamma)\,\zeta]\}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}X_1\,\mathrm{d}Y_1\,\mathrm{d}Z_1. \tag{3·19}$$

The limits for these integrations are discussed in appendix B. The function $\mu^2(z_0)$ has been taken outside the integral sign since it is assumed to be constant over a distance equal at least to that of an elementary layer of thickness L.

The integrations in (3·19) are performed in appendix B to give

$$\langle FF^*\rangle = E_0^2 k^2 \mu^2 L \sec^2\theta' \pi^{\frac{3}{2}} r_0^2 r_z \times\exp\{-\tfrac{1}{4}r_0^2 k^2(\sin^2\theta+\sin^2\theta'-2\sin\theta\sin\theta'\cos\phi_0)-\tfrac{1}{4}r_z^2(\alpha+\gamma)^2\}. \tag{3·20}$$

Now the assembly average of the power crossing unit area perpendicular to the z direction, for component plane waves with their wave normals in the small ranges θ to $\theta+\mathrm{d}\theta$, ϕ to $\phi+\mathrm{d}\theta$ is, from (3·13)

$$\frac{\langle FF^*\rangle}{8\pi^2 Z_0}k^2\sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{3·21}$$

where $k^2\sin\theta\cos\theta$ is the Jacobian $\partial(k_x, k_y)/\partial(\theta,\phi)$. The following change of variable is now made:

$$\sin^2\theta = t. \tag{3·22}$$

Then the ratio of (3·21) to the power flux of the incident wave in the z direction $E_0^2/2Z_0$ is

$$\langle FF^*\rangle \frac{k^2\,\mathrm{d}t\,\mathrm{d}\phi}{8\pi^2E_0^2} = \frac{1}{8(1-t')}k^4\mu^2L\pi^{-\frac{1}{2}}r_0^2r_z\exp\{-\tfrac{1}{4}r_0^2k^2[t+t'-2(tt')^{\frac{1}{2}}\cos\phi_0]-\tfrac{1}{4}r_z^2(\alpha+\gamma)^2\}\,\mathrm{d}t\,\mathrm{d}\phi$$

$$= \beta Lp(\theta',\phi';\theta,\phi)\sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{3·23}$$

say, where β is independent of θ, ϕ, and the function p is chosen so that

$$\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} p\sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 1. \tag{3·24}$$

If (3·23) is integrated with respect to t (equivalently θ) and ϕ, it gives the ratio of the total scattered power to the incident power flux for a slab of thickness L. Thus $\beta(\theta',\phi')$ may be found. It will be called the 'coefficient for total scattering'. It is shown in the next section that β may be taken as constant in cases of practical interest. Some properties of $p(\theta',\phi';\theta,\phi)$ are also discussed in the next section.

The above formulae have been derived on the assumption that the incident wave is a single plane wave with normal in the direction θ', ϕ'. The argument is still valid, however, if the incident wave is part of an angular spectrum of plane waves consisting of those plane wave components with their wave normals in the small ranges $\mathrm{d}\theta'$, $\mathrm{d}\phi'$ near θ', ϕ'.

4. Small angle approximations

The restrictions so far placed on the angles concerned must be carefully noted. In calculating the field of the waves scattered from an elementary layer it is required that only waves scattered through small angles contribute to the scattered wave. This means that the angle between the wave normals of the incident and scattered waves must be small. The angles θ', θ may, however, both be large and the expression (3·23) is still valid. In interpreting (3·13), (3·21) and (3·23) as power crossing unit area perpendicular to the z direction, it was assumed that θ and θ' are both small. Some further consequences of this assumption will now be examined.

When the change of variable (3·22) is made, (3·12) may be written

$$\alpha+\gamma = \tfrac{1}{2}k[(t-t')+t'\{t\cos^2\phi_0-2(tt')^{\frac{1}{2}}\cos\phi_0+\tfrac{1}{2}(t+t')\}+\ldots]. \tag{4·1}$$

Now t and t' are small so that powers and products greater than the first may be neglected. If the remaining term $\frac{1}{2}k(t-t')$ is inserted in (3·23) it gives

$$\langle FF^*\rangle = 4\pi^2\beta Lp(\theta',\phi';\theta,\phi)\,E_0^2/k^2$$

$$= E_0^2k^2\mu^2L\pi^{\frac{3}{2}}r_0^2r_z\exp[-\tfrac{1}{4}r_0^2k^2\{t+t'-2(tt')^{\frac{1}{2}}\cos\phi_0\}-\tfrac{1}{16}r_z^2k^2(t-t')^2], \tag{4·2}$$

where t' in the denominator of (3·23) has been neglected in comparison with unity.

Similar results have been obtained by other authors, most of whom make the further approximation $t=\sin^2\theta\approx\theta^2$ in the exponent. Thus Fejer (1953) gave an expression which is the same as (4·2) (with θ^2 for t, θ'^2 for t') except that the last exponent involving r_z^2 is absent, so that the effect of correlation of the irregularities in the z direction does not appear.

Bowhill (1961 *b*) pointed this out and used a completely different method to derive the angular spectrum of the scattered wave for a normally incident plane wave. If t' is set equal to zero in (4·2) it gives Bowhill's result (with θ^2 for t). The effect of r_z in (4·2), noted by Bowhill (1961 *b*), is discussed later in § 11.

The total scattered power flux for a normally incident wave may be found by integrating (3·21) over all angles, after making the change of variable (3·22) and setting in $\langle FF^* \rangle$ as given by (4·2) with $t' = 0$. Thus

$$\frac{\langle EE^* \rangle_{\text{tot.}}}{2Z_0} = \frac{E_0^2 k^3}{4Z_0} \mu^2 L \pi r_0^2 \exp(q^2)(1 - \operatorname{erf} q), \tag{4·3}$$

where
$$q = kr_0^2/2r_z. \tag{4·4}$$

This result may also be derived directly from (3·8) by setting $\theta' = 0$, introducing the auto-correlation function (3·15), (3·17) and integrating through the scattering layer, which provides a check (Uscinski 1967).

Use of (3·21), (3·23) and the property of p given in (3·24) leads to

$$\frac{\langle EE^* \rangle_{\text{tot.}}}{2Z_0} = \beta L E_0^2/2Z_0 \tag{4·5}$$

and so from (4·3) the 'coefficient for total scattering' for a normally incident wave is

$$\beta(0) = \tfrac{1}{2}\mu^2 k^3 \pi r_0^2 \exp(q^2)\{1 - \operatorname{erf} q\}. \tag{4·6}$$

The properties of $p(0)$ are discussed by Uscinski (1967) who showed that in the majority of cases of interest, where the irregularities are not greatly lengthened in the direction of wave propagation, $q \gg 1$ and the 'coefficient of total scattering' has the asymptotic value

$$\beta(0) \approx \mu^2 k^2 \surd \pi \, r_z. \tag{4·7}$$

The 'coefficient for total scattering' for arbitrary angle of incidence is obtainable in principle by integrating (4·2) with respect to ϕ_0, t. While the ϕ_0 integration may readily be performed, it is more difficult to carry out the integration with respect to t. If we assume that r_z is not very much greater than r_0, the term involving r_z may be omitted from the exponent of (4·2) which may then be integrated with respect to t, to give

$$\beta(\theta') = \beta(0) \tag{4·8}$$

which is independent of the angle of incidence for small angles.

On the other hand if $r_z \gg r_0$ then the term involving r_0 may be omitted and (4·2) integrated to give

$$\beta(\theta') = \tfrac{1}{2}k^3 \mu^2 \pi r_0^2 [1 - \operatorname{erf}(\tfrac{1}{4} r_z k \theta'^2)] \tag{4·9}$$

which is not independent of the angle of incidence except for extremely small θ'.

In the remainder of the paper, which deals with multiple scatter, it will be assumed that the maximum value of the axial ratio r_z/r_0 is less than about 20. In view of the restriction (*a*) of § 1 this means that $\beta(\theta') \approx \beta(0)$, and enables the effect on the angular spectrum of multiply scattered radiation, for axial ratios up to 20, to be examined using the methods of this paper. If this value for β is set in (4·2) the following expression is obtained

$$p(\theta', \phi'; \theta, \phi) = \frac{a^2}{\pi} \exp[-a^2\{t + t' - 2(tt')^{\frac{1}{2}} \cos\phi_0\} - b^2(t - t')^2], \tag{4·10}$$

where $$a^2 = r_0^2k^2/4, \quad b^2 = r_z^2k^2/16. \tag{4·11}$$

This is the form for $p(\theta', \phi', \theta, \phi)$ which will be used later on in § 6 in the general equations for multiple scatter.

5. Scattering of power by a thin layer

The processes occurring when power incident on an elementary layer is scattered will now be examined. This is necessary in order to determine how the quantities S_1, S_2 and K, defined by (2·1), change when the waves are re-scattered by the medium. This allows the single scatter treatment to be extended to deal with the case of multiple scatter.

Let a very thin phase changing screen be situated at the plane $z = z'$ and suppose that the phase change $\Delta\Phi$ which it produces includes a spatial Fourier component for which

$$\Delta\Phi = \Delta\Phi_0 \cos(kx\sin\theta - \epsilon). \tag{5·1}$$

This is like the screen considered by Ratcliffe (1956, § 3·4) who discussed only the case $\epsilon = 0$. Here the spatial phase ϵ is different for different members of the assembly. The factor $k\sin\theta$ in (5·1) is the same as Ratcliffe's factor $2\pi/d$.

Suppose that a plane wave with field $E = \mathrm{e}^{-\mathrm{i}kz}$ is incident on this screen. Then Ratcliffe's method shows that the scattered wave just below the screen includes a contribution

$$\exp(-\mathrm{i}kz')\exp\{-\mathrm{i}\Delta\Phi_0\cos(kx\sin\theta-\epsilon)\} \tag{5·2}$$

or, since $\Delta\Phi_0$ is extremely small

$$\exp(-\mathrm{i}kz')[1-\mathrm{i}\Delta\Phi_0\cos(kx\sin\theta-\epsilon)]. \tag{5·3}$$

Here the first term is the unscattered incident wave, and the remaining term is the scattered wave, composed of two oblique plane waves whose total field is in quadrature with the incident wave. For other values of z beyond the screen $(z > z')$ the fields (5·3) become

$$\exp(-\mathrm{i}kz)[1-\mathrm{i}\Delta\Phi_0\exp\{-\mathrm{i}k(z-z')(\cos\theta-1)\}\cos(kx\sin\theta-\epsilon)]. \tag{5·4}$$

Here the factor $\exp\{-\mathrm{i}k(z-z')(\cos\theta-1)\}$ occurs because of the different phase velocities in the z direction of the incident wave and the pair of oblique scattered waves. The separate component plane waves in the scattered field are found by expressing the last cosine in (5·4) as the sum of two exponentials. One of them is given by

$$-\tfrac{1}{2}\mathrm{i}\Delta\Phi_0\exp\{-\mathrm{i}kz'\}\exp\{-\mathrm{i}k[(z-z')\cos\theta+x\sin\theta]+\mathrm{i}\epsilon\} \tag{5·5}$$

and the other by reversing the signs of $\sin\theta$ and ϵ in (5·5).

In the single wave (5·5) the phase depends upon x, and so although the total power flux $\sigma_1+\sigma_2 = P$ is constant, the co-phased and quadrature powers depend on x. For an assembly of screens, with a random distribution of the spatial phase ϵ, the assembly averages would give $\sigma_1 = \sigma_2 = \frac{1}{2}P$, $\kappa = 0$, and these are independent of x. When the second wave is added to (5·5) the resultant co-phased power at the screen is zero and the assembly averages give $\sigma_2 = 2P$, $\sigma_1 = \kappa = 0$. Thus it is permissible to treat the two waves as though they separately have $\sigma_2 = P$, $\sigma_1 = \kappa = 0$ at the scattering layer.

This result applies more generally to a set $\mathscr{A}$ of component scattered waves with their wave normals in the range θ to $\theta+\mathrm{d}\theta$, ϕ to $\phi+\mathrm{d}\phi$. If they were produced by the scattering of a normally incident wave, then there must also be present another set $\mathscr{B}$ of components of equal amplitude, with wave normals in the range θ to $\theta+\mathrm{d}\theta$, $\phi+\pi$ to $\phi+\mathrm{d}\phi+\pi$. Each of these sets may be taken as having $\sigma_1 = \kappa = 0$ and equal values of σ_2. When, later on, the integration with respect to ϕ is done, these two values of σ_2 are added together to give the correct total σ_2.

The set $\mathscr{A}$ of components might, however, be produced by second scattering of another set $\mathscr{C}$ of component plane waves with wave normals in the range θ' to $\theta'+\mathrm{d}\theta'$ and ϕ' to $\phi'+\mathrm{d}\phi'$, which were themselves produced by the scattering of a normally incident wave. There must then be a set $\mathscr{D}$ with normals in the range θ' to $\theta'+\mathrm{d}\theta'$ and $\phi'+\pi$ to $\phi'+\mathrm{d}\phi'+\pi$ equal in amplitude to $\mathscr{C}$. The resultant of $\mathscr{C}$ and $\mathscr{D}$ is in quadrature with the incident wave.

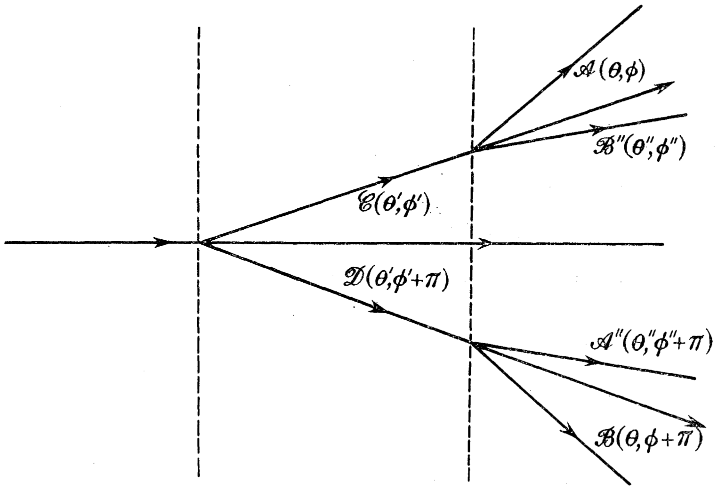

FIGURE 3. Formation of corresponding wave sets when one wave set is scattered.

Second scattering of $\mathscr{D}$ would then produce the set $\mathscr{B}$ described above and the resultant of $\mathscr{A}$ and $\mathscr{B}$ is in quadrature with the resultant of $\mathscr{C}$ and $\mathscr{D}$. Thus by an extension of the argument for single scatter it is permissible to treat the set $\mathscr{A}$ as though it were in quadrature with the set $\mathscr{C}$ from which it was produced by scattering. The argument can clearly be extended to deal with waves which are scattered three or more times.

Of course scattering of the set $\mathscr{C}$ or $\mathscr{D}$ produces other sets like $\mathscr{A}$, with components in all angles θ, ϕ. So far attention has been directed only to the specific set $\mathscr{A}$ with normals in the range θ to $\theta+\mathrm{d}\theta$ and ϕ to $\phi+\mathrm{d}\phi$. Of the other sets like $\mathscr{A}$ there would be one, say $\mathscr{A}''$, with normals in the range θ'' to $\theta''+\mathrm{d}\theta''$, $\phi''+\pi$ to $\phi''+\pi+\mathrm{d}\phi''$, and of the other sets like $\mathscr{B}$ there would be one, say $\mathscr{B}''$, with normals in the range θ'' to $\theta''+\mathrm{d}\theta''$, ϕ'' to $\phi''+\mathrm{d}\phi''$ (see figure 3). The resultant of $\mathscr{A}''$ and $\mathscr{B}''$ immediately after scattering is in quadrature with the resultant of $\mathscr{C}$ and $\mathscr{D}$ immediately before scattering. Thus the resultant of the pair $\mathscr{A}$, $\mathscr{B}$ and of the pair $\mathscr{A}''$, $\mathscr{B}''$ start out from the scattering layer in phase with each other. If the angle θ of the set $\mathscr{A}$, $\mathscr{B}$ is not the same as the angle θ'' of the set $\mathscr{A}''$, $\mathscr{B}''$, the phase of these resultants will differ when they have travelled some distance from the scattering layer.

The use of these concepts makes it possible to study how S_1, S_2 and K, the co-phased and quadrature scattered power and the un-normalized correlation coefficient of the scattered power, behave when the waves are re-scattered by an elementary layer. Consider the scattered field E_1, with co-phased and quadrature components R and I respectively, to be incident on a scattering layer. On passing through the layer E_1 acquires an increment δE_1 whose co-phased and quadrature components may be written δR and δI respectively. Then from (2·1)

$$\left.\begin{aligned} \delta S_1 &= 2\langle R\,\delta R\rangle + \langle \delta R^2\rangle, \\ \delta S_2 &= 2\langle I\,\delta I\rangle + \langle \delta I^2\rangle, \\ \delta K &= \langle I\,\delta R\rangle + \langle R\,\delta I\rangle + \langle \delta I\,\delta R\rangle. \end{aligned}\right\} \tag{5·6}$$

Since the incremental field may be taken to be in quadrature with the field of the incident wave at the scattering layer, an incident R gives rise to δI, and an incident I gives rise to δR. This may be indicated symbolically by writing

$$\left.\begin{aligned} \delta I &= MR, \\ \delta R &= -MI, \end{aligned}\right\} \tag{5·7}$$

where M is an integral operator similar to that on the right-hand side of (3·8).

Assembly averages with one δ, e.g. $\langle R\,\delta I\rangle$, involve the assembly average of M whose integrand contains n_1 to the first power. But the assembly average of n_1 is zero, and so all products in (5·6) which contain only one δ term give zero. Thus (5·6) becomes

$$\delta S_1 = M^2 S_2, \quad \delta S_2 = M^2 S_1, \quad \delta K = -M^2 K. \tag{5·8}$$

These relations are still valid if S_1, S_2 and K refer only to that part of the incident wave whose component plane waves are the set $\mathscr{C}$ with normals in the range θ' to $\theta'+\mathrm{d}\theta'$ and ϕ' to $\phi'+\mathrm{d}\phi'$, while δS_1, δS_2 and δK refer to that part of the scattered wave whose component waves are the set $\mathscr{A}$ with normals in the range θ to $\theta+\mathrm{d}\theta$ and ϕ to $\phi+\mathrm{d}\phi$. In this case the integral operator is

$$\beta Lp(\theta', \phi'; \theta, \phi) \cos\theta$$

as given by (3·23), and the relations (5·8) become, since $\cos\theta \approx 1$,

$$\left.\begin{aligned} \delta\sigma_1(z,\theta,\phi) &= \beta Lp(\theta',\phi';\theta,\phi)\,\sigma_2(z,\theta',\phi'), \\ \delta\sigma_2(z,\theta,\phi) &= \beta Lp(\theta',\phi';\theta,\phi)\,\sigma_1(z,\theta',\phi'), \\ \delta\kappa(z,\theta,\phi) &= -\beta Lp(\theta',\phi';\theta,\phi)\,\kappa(z,\theta',\phi'). \end{aligned}\right\} \tag{5·9}$$

6. Scattering by an extended medium

Suppose that a plane wave is normally incident on a medium consisting of many elementary layers of the kind considered in § 3. Let its electric field be

$$E(z)\,\mathrm{e}^{-ikz}. \tag{6·1}$$

The object is now to find $S_1(z)$, $S_2(z)$, $K(z)$ and the unscattered power $|E(z)|^2$ of the incident wave when a distance z has been traversed. It is assumed that each elementary layer is too thin for multiple scatter to be important within it, but it scatters both the incident wave which reaches it, and also waves which have previously been scattered by other elementary layers. Thus S_1, S_2, K change as the medium is traversed. Their rates of change will be found by deriving first the rates of change of their angular spectrum components (2·5), namely

$$\frac{\partial}{\partial z}\sigma_1(z,\theta,\phi), \quad \frac{\partial}{\partial z}\sigma_2(z,\theta,\phi), \quad \frac{\partial}{\partial z}\kappa(z,\theta,\phi) \tag{6·2}$$

thus giving differential equations which can be solved by stepwise integration proceeding through the medium. The solutions may then be integrated over θ and ϕ to give $S_1(z)$, $S_2(z)$, $K(z)$.

There are four mechanisms contributing to the rates of change (6·2).

(a) Scattering of the incident wave

When the incident wave (6·1) first enters the medium, at $z = 0$, its power flux is

$$\Pi(0) = E_0^2(0)/2Z_0. \tag{6·3}$$

Power is removed from it by scattering at the rate given by (4·5) with $L = \mathrm{d}z$ so that

$$\partial\Pi(z)/\partial z = -\beta\Pi(z). \tag{6·4}$$

Thus the power flux $\Pi(z)$ of the incident wave at any z is

$$\Pi(z) = \Pi(0)\exp(-S), \quad S = \int_0^z \beta(z)\,\mathrm{d}z. \tag{6·5}$$

The scattered power is distributed as an angular spectrum and contributes to the rates of change (6·2). This contribution may be found by using (5·9), and (6·5) (with $L = \mathrm{d}z$) since these relations, which were derived for the general case of an already scattered field, are valid for the unscattered incident wave also. In this case, however, because the incident wave itself is taken as phase reference, all the unscattered power arriving at any layer is co-phased and has no quadrature component, and its K is zero. Thus in the absence of other effects

$$\frac{\partial\sigma_1}{\partial z}(z,\theta,\phi) = \frac{\partial\kappa}{\partial z}(z,\theta,\phi) = 0 \tag{6·6}$$

and

$$\frac{\partial\sigma_2}{\partial z}(z,\theta,\phi) = \beta p(\theta,\phi;0)\,\Pi(z). \tag{6·7}$$

(b) Removal of power by scattering

When the power flux in the small angular range θ to $\theta+\mathrm{d}\theta$, ϕ to $\phi+\mathrm{d}\phi$ impinges on an elementary layer, a small amount of it is scattered. The ratio of total scattered power flux to incident power flux is given from (3·23) by $\beta\,\mathrm{d}z$. This applies for both the co-phased and quadrature power fluxes σ_1 and σ_2, as well as for the un-normalized correlation coefficient κ. If this were the only effect it would give

$$\frac{\partial\sigma_1}{\partial z} = -\beta\sigma_1, \quad \frac{\partial\sigma_2}{\partial z} = -\beta\sigma_2, \quad \frac{\partial\kappa}{\partial z} = -\beta\kappa. \tag{6·8}$$

(c) The distance effect

The scattered power is treated as a spectrum of plane waves travelling at different angles. It was shown in § 5 that for every wave with its normal in the direction θ, ϕ there is another wave of equal amplitude with its normal in the direction θ, $\phi+\pi$, so that the phase of the resultant does not depend on x and y (except for a plus or minus sign which does not affect the quantities σ_1, σ_2, κ, as can be seen by setting $\chi = \chi\pm\pi$ in (2·2)), but only on z, see (5·4).

The phase reference is the unscattered wave e^{-ikz}. The phase of a scattered wave, travelling obliquely, changes with respect to this reference as z increases. The factor

$\exp\{-ik(z-z')(\cos\theta-1)\}$ in (5·4) gives the change in phase of the scattered wave relative to that of e^{-ikz} in advancing from z' to z. Thus in advancing this distance the scattered wave leads the phase reference by a phase angle:

$$\chi = k(1-\cos\theta)(z-z'). \tag{6·9}$$

If σ_1', σ_2', κ' denote the values of σ_1, σ_2, κ where $z = z'$ for the wave (5·5) and its partner, and if σ_1, σ_2, κ are their values for any other z, then (2·2) gives

$$\left.\begin{aligned}\sigma_1 &= \sigma_1'\cos^2\chi+\sigma_2'\sin^2\chi-2\kappa'\sin\chi\cos\chi,\\ \sigma_2 &= \sigma_1'\sin^2\chi+\sigma_2'\cos^2\chi+2\kappa'\sin\chi\cos\chi,\\ \kappa &= -(\sigma_2'-\sigma_1')\sin\chi\cos\chi+\kappa'(\cos^2\chi-\sin^2\chi).\end{aligned}\right\} \tag{6·10}$$

Differentiation of (6·10) with respect to z, and use of (6·9) to find $\partial\chi/\partial z$ gives

$$\left.\begin{aligned}\partial\sigma_1/\partial z &= -2\kappa k(1-\cos\theta),\\ \partial\sigma_2/\partial z &= 2\kappa k(1-\cos\theta),\\ \partial\kappa/\partial z &= -(\sigma_2-\sigma_1)k(1-\cos\theta).\end{aligned}\right\} \tag{6·11}$$

In the absence of other effects, equations (6·11) give the rates of change (6·2) due to the fact that the scattered radiation is travelling at an angle to the incident plane wave which is the phase reference.

(*d*) *Multiple scatter*

Finally, increments of σ_1, σ_2 and κ may be produced by scattering of waves which have been scattered one or more times. The component waves impinging on an elementary layer with wave normals in the solid angle $d\Omega' = \sin\theta'\,d\theta'\,d\phi'$, i.e. in the range θ' to $\theta'+d\theta'$, ϕ' to $\phi'+d\phi'$, give increments which may be obtained from (5·9). When integrated with respect to $d\Omega'$ these give the contribution due to multiple scatter from all angles:

$$\left.\begin{aligned}\partial\sigma_1(z,\theta,\phi)/\partial z &= \beta\iint p(\theta',\phi';\theta,\phi)\,\sigma_2(z,\theta',\phi')\,d\Omega',\\ \partial\sigma_2(z,\theta,\phi)/\partial z &= \beta\iint p(\theta',\phi';\theta,\phi)\,\sigma_1(z,\theta',\phi')\,d\Omega',\\ \partial\kappa(z,\theta,\phi)/\partial z &= -\beta\iint p(\theta',\phi';\theta,\phi)\,\kappa(z,\theta',\phi')\,d\Omega'.\end{aligned}\right\} \tag{6·12}$$

7. Discussion of the equations

The four contributions of § 6, equations (6·6)–(6·8), (6·11), (6·12), are now added to give

$$\left.\begin{aligned}\left(\frac{\partial}{\partial z}+\beta\right)\sigma_1(z,\theta,\phi) &= 2\kappa(z,\theta,\phi)\,k(\cos\theta-1)+\beta\iint p(\theta',\phi';\theta,\phi)\,\sigma_2(z,\theta',\phi')\,d\Omega',\\ \left(\frac{\partial}{\partial z}+\beta\right)\sigma_2(z,\theta,\phi) &= \beta p(\theta,\phi;0)\,\Pi(z)-2\kappa(z,\theta,\phi)\,k(\cos\theta-1)\\ &\qquad +\beta\iint p(\theta',\phi';\theta,\phi)\sigma_1(z,\theta',\phi')\,d\Omega',\\ \left(\frac{\partial}{\partial z}+\beta\right)\kappa(z,\theta,\phi) &= \{\sigma_2(z,\theta,\phi)-\sigma_1(z,\theta,\phi)\}\,k(\cos\theta-1)\\ &\qquad -\beta\iint p(\theta',\phi';\theta,\phi)\,\kappa(z,\theta',\phi')\,d\Omega'.\end{aligned}\right\} \tag{7·1}$$

These equations are completely general. They are now simplified by changes of notation and by making special assumptions as follows:

1. The medium is statistically isotropic with respect to rotation about the z axis. Therefor σ_1, σ_2 and κ are independent of ϕ.

2. The function $p(\theta, \phi, 0)$ is independent of ϕ which will be omitted. The function $p(\theta', \phi'; \theta, \phi)$ depends only on the difference $\phi' - \phi$ and when the ϕ' integration is performed in (7·1) the result is independent of ϕ. Thus ϕ' and ϕ will be omitted from the bracket.

3. It is assumed that β is independent of z so that (6·5) gives

$$\Pi(z) = \Pi(0)\, e^{-\beta z}.$$

Further, the three left-hand sides of (7·1) are each of the form $e^{-\beta z}\, \partial(U e^{\beta z})/\partial z$ where U denotes σ_1, σ_2 or κ. This suggests the use of new variables $(\sigma_1, \sigma_2, \kappa)\, e^{\beta z}$.

4. Some shortening is achieved by using the combinations $\sigma_2 + \sigma_1$, $\sigma_2 - \sigma_1$, 2κ instead of σ_1, σ_2 and κ. The following new variables are therefore introduced

$$\left.\begin{aligned} P_1(z,\theta) &= \{\sigma_2(z,\theta,\phi) + \sigma_1(z,\theta,\phi)\}\, e^{\beta z}, \\ P_2(z,\theta) &= \{\sigma_2(z,\theta,\phi) - \sigma_1(z,\theta,\phi)\}\, e^{\beta z}, \\ P_3(z,\theta) &= 2\kappa(z,\theta,\phi)\, e^{\beta z}. \end{aligned}\right\} \tag{7·2}$$

Then from (7·1), on adding and subtracting the first two equations,

$$\partial P_1(z,\theta)/\partial z = \beta p(\theta, 0)\, \Pi(0) + \beta \iint p(\theta', \theta)\, P_1(z, \theta')\, d\Omega', \tag{7·3}$$

$$\partial P_2(z,\theta)/\partial z = \beta p(\theta, 0)\, \Pi(0) - 2P_3(z,\theta)\, k(\cos\theta - 1) - \beta \iint p(\theta', \theta)\, P_2(z, \theta')\, d\Omega', \tag{7·4}$$

$$\partial P_3(z,\theta)/\partial z = 2P_2(z,\theta)\, k(\cos\theta - 1) - \beta \iint p(\theta', \theta)\, P_3(z, \theta')\, d\Omega'. \tag{7·5}$$

Particular forms for the functions $p(\theta, 0)$, $p(\theta, \theta')$ derived in § 4, equation (4·10) may now be introduced and the following changes of variable made

$$l = \beta z, \quad \tau = a^2 t. \tag{7·6}$$

Since the case where the total angle of scatter is small is being considered then

$$(\cos\theta - 1) \approx -\tfrac{1}{2}t \tag{7·7}$$

and for convenience the incident power may be given the value

$$\Pi(0) = E_0^2(0)/2Z_0 = 1/a^2. \tag{7·8}$$

This latter step is permissible since $\Pi(0)$ may be measured in arbitrary units.

When equations (7·3)–(7·5) are thus simplified and the integration over ϕ' performed in the terms on the right-hand side, the following set of equations results:

$$\frac{\partial P_1}{\partial l}(l,\tau) = \exp(-\tau - D\tau^2) + \exp(-\tau - D\tau^2) \times \int_0^\infty \exp(-\tau' - D[\tau'^2 - 2\tau\tau'])\, P_1(l,\tau')\, I_0\{2(\tau\tau')^{\frac{1}{2}}\}\, d\tau', \tag{7·9}$$

$$\frac{\partial P_2}{\partial l}(l,\tau) = \exp(-\tau - D\tau^2) + A\tau P_3(l,\tau) - \exp(-\tau - D\tau^2) \times \int_0^\infty \exp(-\tau' - D[\tau'^2 - 2\tau\tau'])\, P_2(l,\tau')\, I_0\{2(\tau\tau')^{\frac{1}{2}}\}\, d\tau', \tag{7·10}$$

$$\frac{\partial P_3}{\partial l}(l,\tau) = -A\tau P_2(l,\tau) - \exp\left(-\tau - D\tau^2\right) \times \int_0^\infty \exp\left(-\tau' - D[\tau'^2 - 2\tau\tau']\right) P_3(l,\tau')\, I_0\{2(\tau\tau')^{\frac{1}{2}}\}\, d\tau' \quad (7\cdot11)$$

(I_0 denotes a modified Bessel function).

It is interesting to note that these equations contain only two parameters,

$$D = (r_z\lambda/2\pi r_0^2)^2 = 1/4q^2 \quad (7\cdot12)$$

and

$$A = 2k(1-\cos\theta)/\beta\tau = \frac{1}{2\pi^{\frac{7}{2}}}\left(\frac{\lambda^3}{\mu^2 r_0^2 r_z}\right). \quad (7\cdot13)$$

The parameter D determines the effect of the axial ratio r_z/r_0 on the scattering. Since the wavelength of the radiation must be much less than the smallest scale size of the irregularities, and since the largest axial ratio dealt with here is assumed to be less than about 20, D is restricted to values less than 0·1. The effect of D on the scattering will be examined in detail in § 11, where it is shown that for values of less than 0·1, D may virtually be neglected and equations (7·9)–(7·11) thus simplified.

In cases of practical interest equations (7·9)–(7·11) contain only one parameter A. This parameter, which couples equations (7·10), (7·11), is a measure of the relative importance of multiple scatter (6*d*) to the distance effect (6*c*) as mechanisms determining the passage of the scattered power from quadrature to co-phased power over the region of interest. If A approaches zero, which may result from large r_0 or r_z, the angle through which the radiation is scattered is very small indeed, as may be seen from (4·10). Since it is travelling virtually parallel to the incident wave which is the phase reference, a scattered wave in the angular spectrum does not experience any significant phase shift relative to the reference until it has covered a very large distance, and so the scattered power tends to remain in quadrature. However, in travelling this distance it may be re-scattered many times, especially if μ^2 is not excessively small, thus producing a co-phased scattered component. If A is very small then multiple scatter is the important mechanism leading to the equalization of co-phased and quadrature scattered power.

If A is large, which results from small r_0, r_z, μ^2, then the scattered power is spread over a range of angles which is not small, as may be seen from (4·10), and the phase of the scattered wave changes rapidly relative to that of the incident wave as z increases. This means that before significant multiple scatter can take place a co-phased scattered component is produced due to the distance effect alone. Thus for large A the distance effect is the chief mechanism leading to the equalization of co-phased and quadrature scattered power and the whole process may be treated as a case of single scatter.

Analytical solutions for the equations (7·3)–(7·5), (7·9)–(7·11), may be obtained in these two extreme cases noted above, i.e. A very large and very small, in cases of practical interest where D may be neglected.

8. Analytical solutions

Analytical solutions for equations (7·3)–(7·5) may easily be obtained in the extreme cases of A very large and very small if the scattering cross-section given by (4·10) is used and the term involving r_z in the exponent is neglected to give

$$p(\theta;\theta') = a^2\pi^{-1}\exp\left[-a^2\{t+t'-2(tt')^{\frac{1}{2}}\cos\phi_0\}\right]. \quad (8\cdot1)$$

In the small angle approximation this cross-section gives results which are very close to those obtained by using the full form (4·2), provided that r_z is not very much larger than r_0. This is demonstrated by the full numerical solution of equations (7·9)–(7·11) when the r_z term is included, § 11.

Case 1. *A very large*. As pointed out at the end of the previous section this case may be treated as one of single scatter over the region of interest, and the integrals on the right-hand side of equations (7·3)–(7·5) omitted. The resulting equations may readily be solved for P_1, P_2, and P_3, since these three quantities are all zero when $z = 0$.

Then from (7·2)

$$\sigma_1(z,\theta,\phi) = \tfrac{1}{2}\beta\,\mathrm{e}^{-\beta z}p(\theta,0)\,\Pi(0)\,z[1-\sin(2\chi_0)/2\chi_0], \tag{8·2}$$

$$\sigma_2(z,\theta,\phi) = \tfrac{1}{2}\beta\,\mathrm{e}^{-\beta z}p(\theta,0)\,\Pi(0)\,z[1+\sin(2\chi_0)/2\chi_0], \tag{8·3}$$

$$\kappa(z,\theta,\phi) = \tfrac{1}{2}\beta\,\mathrm{e}^{-\beta z}p(\theta,0)\,\Pi(0)\,z[(1-\cos 2\chi_0)/2\chi_0], \tag{8·4}$$

where

$$\chi_0 = kz(1-\cos\theta). \tag{8·5}$$

If (8·1) is introduced into equations (8·2)–(8·4) they may be integrated over θ and ϕ to give

$$S_1(z) = \tfrac{1}{2}\beta\,\mathrm{e}^{-\beta z}\,\Pi(0)\,z\left[1-\frac{1}{2Bz}\arctan(2Bz)\right], \tag{8·6}$$

$$S_2(z) = \tfrac{1}{2}\beta\,\mathrm{e}^{-\beta z}\,\Pi(0)\,z\left[1+\frac{1}{2Bz}\arctan(2Bz)\right], \tag{8·7}$$

$$K(z) = -\tfrac{1}{2}\beta\,\mathrm{e}^{-\beta z}\,\Pi(0)\,z\frac{1}{4Bz}\ln[1+(2Bz)^2], \tag{8·8}$$

where

$$B = 2/kr_0^2. \tag{8·9}$$

These results can be derived directly from equation (3·8) by other methods for the special case $\theta' = 0$, which provides a check. This has been done in another paper by Uscinski (1967) who gives a full discussion of the results for the single scatter case. Similar results were obtained by Obukhov (1953) and Chernov (1960) using the 'method of smooth perturbations'. The extra factor $\exp(-\beta z)$ in expressions (8·6)–(8·8) occurs because the attenuation of the unscattered wave has been taken into account in the present paper.

Case 2. *A very small*. The appropriate equations for this case may be obtained by setting both A and D equal to zero in (7·9)–(7·11). Exact solutions of the resulting equations may be obtained by successive approximation. A first approximation to the solution is obtained by neglecting the integral terms and solving the resulting equations. This approximate solution is set into the integral and the equations solved again to give a better approximation. The process is repeated and solutions in the form of infinite series are obtained:

$$P_1 = \sum_{j=1}^{\infty}\frac{l^j}{j!\,j}\exp(-\tau/j), \tag{8·10}$$

$$P_2 = -\sum_{j=1}^{\infty}\frac{(-l)^j}{j!\,j}\exp(-\tau/j), \tag{8·11}$$

$$P_3 = 0, \tag{8·12}$$

where τ is given by (7·6). When (7·2), (7·6), (3·22) and (4·11) are used, P_1 may be identified with the angular spectrum of the total scattered power found by Fejer (1953) and Howells (1960) who neglected the correlation of the irregularities in the z direction given by r_z.

The solution (8·12) is to be expected from physical considerations, since for $A = 0$ there is no distance effect and all the co-phased scattered power is just as likely to be in anti-phase as in phase with the incident wave, and the correlation, given by averages of products like that in (2·1), is zero.

Use of (7·2), (7·6) and (3·22) gives

$$\sigma_1(z,\theta,\phi) = \exp(-\beta z) \sum_{j=1}^{\infty} \frac{(\beta z)^{2j-1}}{(2j-1)!\,(2j-1)} \exp\{-a^2 \sin^2\theta/2j-1\}, \tag{8·13}$$

$$\sigma_2(z,\theta,\phi) = \exp(-\beta z) \sum_{j=1}^{\infty} \frac{(\beta z)^{2j}}{(2j)!\,(2j)} \exp(-a^2 \sin^2\theta/2j\}, \tag{8·14}$$

$$\kappa(z,\theta,\phi) = 0. \tag{8·15}$$

These equations may be integrated with respect to θ and ϕ to give

$$\left.\begin{aligned} S_1(z) &= E_0^2\,\mathrm{e}^{-\beta z}(\cosh\beta z - 1), \\ S_2(z) &= E_0^2\,\mathrm{e}^{-\beta z}\sinh\beta z, \\ K(z) &= 0. \end{aligned}\right\} \tag{8·16}$$

Solutions of (7·9)–(7·11) for intermediate values of the parameter A have been obtained numerically and are given in § 10. The analytical solutions given above form bounding curves for the family and afford a partial check on the numerical calculations.

9. Numerical methods

As pointed out in § 7, the parameter D which determines the effect of the axial ratio r_z/r_0 on the scattering, may be neglected when $r_z/r_0 \leqslant 20$, and so equations (7·9)–(7·11) may be simplified to give

$$\frac{\partial P_1}{\partial l}(l,\tau) = \exp(-\tau) + \exp(-\tau)\int_0^{\infty} \exp(-\tau')\,P_1(l,\tau')\,I_0\{2(\tau\tau')^{\frac{1}{2}}\}\,\mathrm{d}\tau', \tag{9·1}$$

$$\frac{\partial P_2}{\partial l}(l,\tau) = \exp(-\tau) + A\tau P_3(l,\tau) - \exp(-\tau)\int_0^{\infty} \exp(-\tau')\,P_2(l,\tau')\,I_0\{2(\tau\tau')^{\frac{1}{2}}\}\,\mathrm{d}\tau', \tag{9·2}$$

$$\frac{\partial P_3}{\partial l}(l,\tau) = -A\tau P_2(l,\tau) - \exp(-\tau)\int_0^{\infty} \exp(-\tau')\,P_3(l,\tau')\,I_0\{2(\tau\tau')^{\frac{1}{2}}\}\,\mathrm{d}\tau'. \tag{9·3}$$

The effect of D will be discussed more fully in § 11, where it will be shown why it may be neglected in cases of practical interest.

For the purposes of numerical solution, the integro-differential equations (9·1)–(9·3) were expressed as a set of simultaneous first-order differential equations. This was achieved by expressing the integrals as sums, using Gaussian quadrature and Gauss–Laguerre quadrature, c.f. Salzer & Zucker (1949) and Abramowitz & Stegun (1964). A set of n discrete values of τ was chosen and the functions $P(\tau)$ were evaluated for these values only. Equations (9·1)–(9·3) then become

$$\frac{\partial P_{1,(i)}}{\partial l} = \exp(-\tau_i) + \exp(-\tau_i)\sum_{j=0}^{n-1} G_{ij} P_{1,(j)}, \tag{9·4}$$

$$\frac{\partial P_{2,(i)}}{\partial l} = \exp(-\tau_i) + A\tau_i P_{3,(i)} - \exp(-\tau_i)\sum_{j=0}^{n-1} G_{ij} P_{2,(j)}, \tag{9·5}$$

$$\frac{\partial P_{3,(i)}}{\partial l} = -A\tau_i P_{2,(i)} - \exp(-\tau_i)\sum_{j=0}^{n-1} G_{ij} P_{3,(j)}, \tag{9·6}$$

where $$P_{r,(i)} = P_r(l, \tau_i), \quad P_{r,(j)} = P_r(l, \tau'_j) \quad (r = 1, 2, 3),$$
$$G_{ij} = w_j\, I_0\{2(\tau_i \tau_j)^{\frac{1}{2}}\}$$

and w_j is the weight corresponding to the point j.

The range of integration in the integrals on the right-hand side of (9·1)–(9·3) was divided into two parts. The points τ_j and weights w_j were chosen so that the integrals could be replaced by a Gaussian quadrature over the first part of the range from 0 to a including the first sixteen points, and by a Gauss–Laguerre quadrature over the second part of the range from a to ∞ covered by the remaining eight points. Thus

$$\int_0^\infty e^{-\tau'} f(\tau')\, d\tau' = \sum_{j=0}^{23} f(\tau_j)\, w_j, \tag{9·7}$$

where $w_j = \exp(-\tau_j)\, g_j$, $j \leqslant 15$ and g_j are Gaussian weights, and $w_j = \exp(-a)\, l_j$, $j \geqslant 16$ and l_j are Gauss–Laguerre weights.

The resulting set of coupled differential equations (9·4)–(9·6) was solved numerically by step-wise integration using the Adams–Bashforth technique. The integration commenced at the boundary of the medium where $l = 0$ and the functions P are all zero, and proceeded as far as necessary through the medium giving the required functions P. This was done for different values of the parameter A, and it was found that as A became larger it was necessary to choose progressively smaller steplengths of l in order to avoid oscillation in the integration routine.

The functions σ_1, σ_2 and κ are obtained from the functions P at the points τ_i, by using (7·2), and may then be conveniently integrated numerically with respect to τ with the same points and weights, to give the required S_1, S_2 and K.

Some partial checks on the numerical integration were made. The functions P_1, P_2 and P_3 for the cases $A = 0$ and A large may be obtained analytically, see equations (8·10)–(8·12) and (8·2)–(8·4). These were compared with the appropriate numerical results and their agreement afforded a check on the correct functioning of the numerical integration of equations (2·1)–(2·3). Similarly the analytical expressions for S_1, S_2 and K, for the cases $A = 0$ and A large, as given by equations (8·16) and (8·6)–(8·8) respectively were compared with the corresponding numerical results and afforded a check on the numerical integration of the functions σ_1, σ_2 and κ with respect to τ.

10. Numerical results

Equations (9·1)–(9·3) were solved numerically for various values of the parameter

$$A = \frac{1}{2\pi^{\frac{3}{2}}}\left(\frac{\lambda^3}{\mu^2 r_0^2 r_z}\right)$$

to give P_1, P_2, and P_3 and thus, from (7·2), σ_1, σ_2 and κ. The functions $\sigma_1(z,\theta)$, $\sigma_2(z,\theta)$ and $\kappa(z,\theta)$ are plotted in figure 4 for some representative values of A at different distances $z = l/\beta$ in the medium.

When σ_1, σ_2 and κ are integrated with respect to θ, they give $S_1(z)$, $S_2(z)$ and $K(z)$ respectively. These functions are shown in figure 5 for some values of the parameter A.

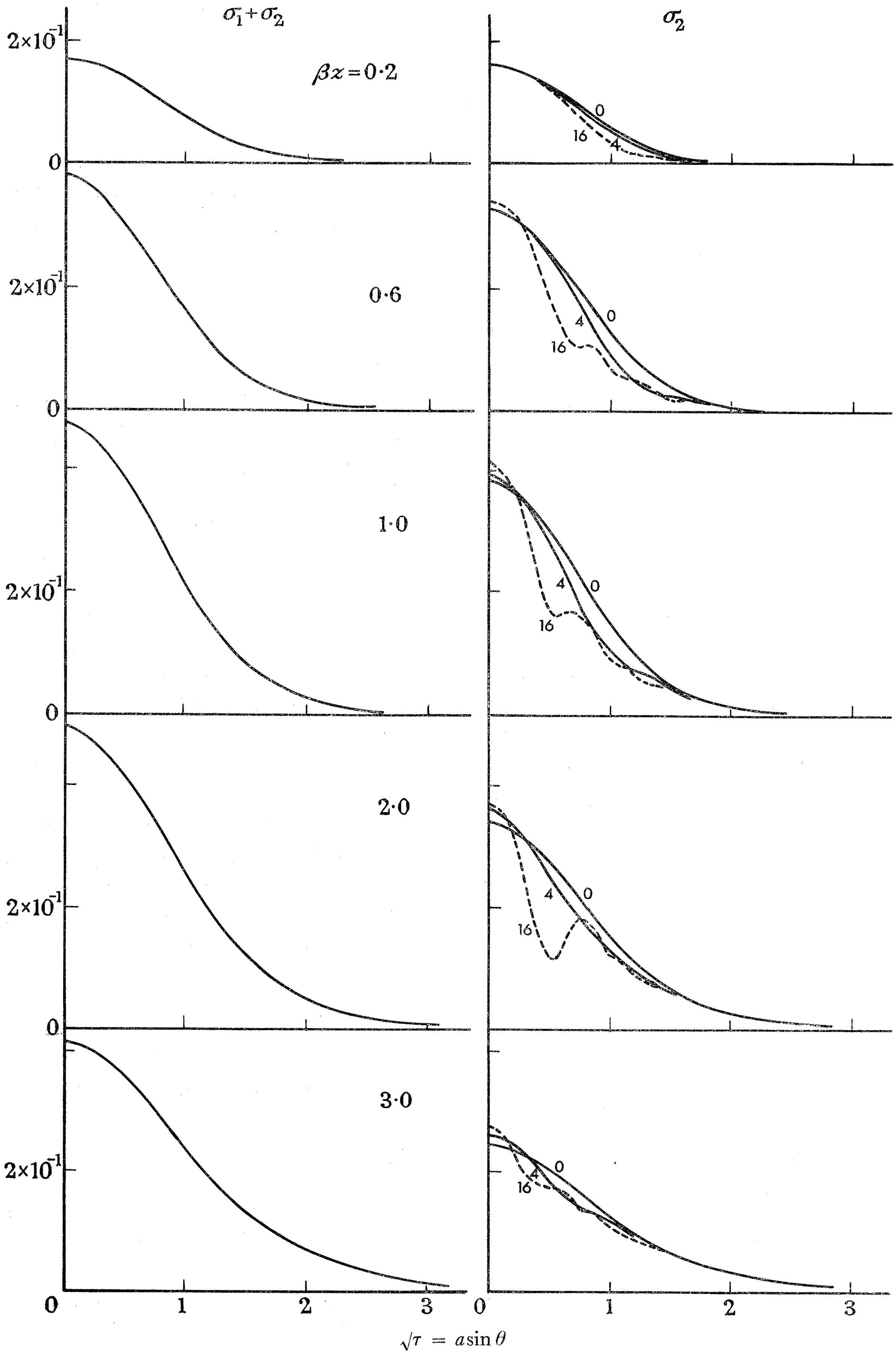

FIGURE 4. The functions $\sigma_1(z, \theta)$, $\sigma_2(z, \theta)$, $\kappa(z, \theta)$ for various values of the parameter A. The horizontal scale is $a \sin\theta$, where $a = \frac{1}{2}r_0 k$. Unity on the vertical scale denotes the value of $\Pi(0)$, the initial power flux of the incident wave. The curve for $A = 16$ is a broken line so that it can be distinguished. For $A = 0$, κ is zero everywhere.

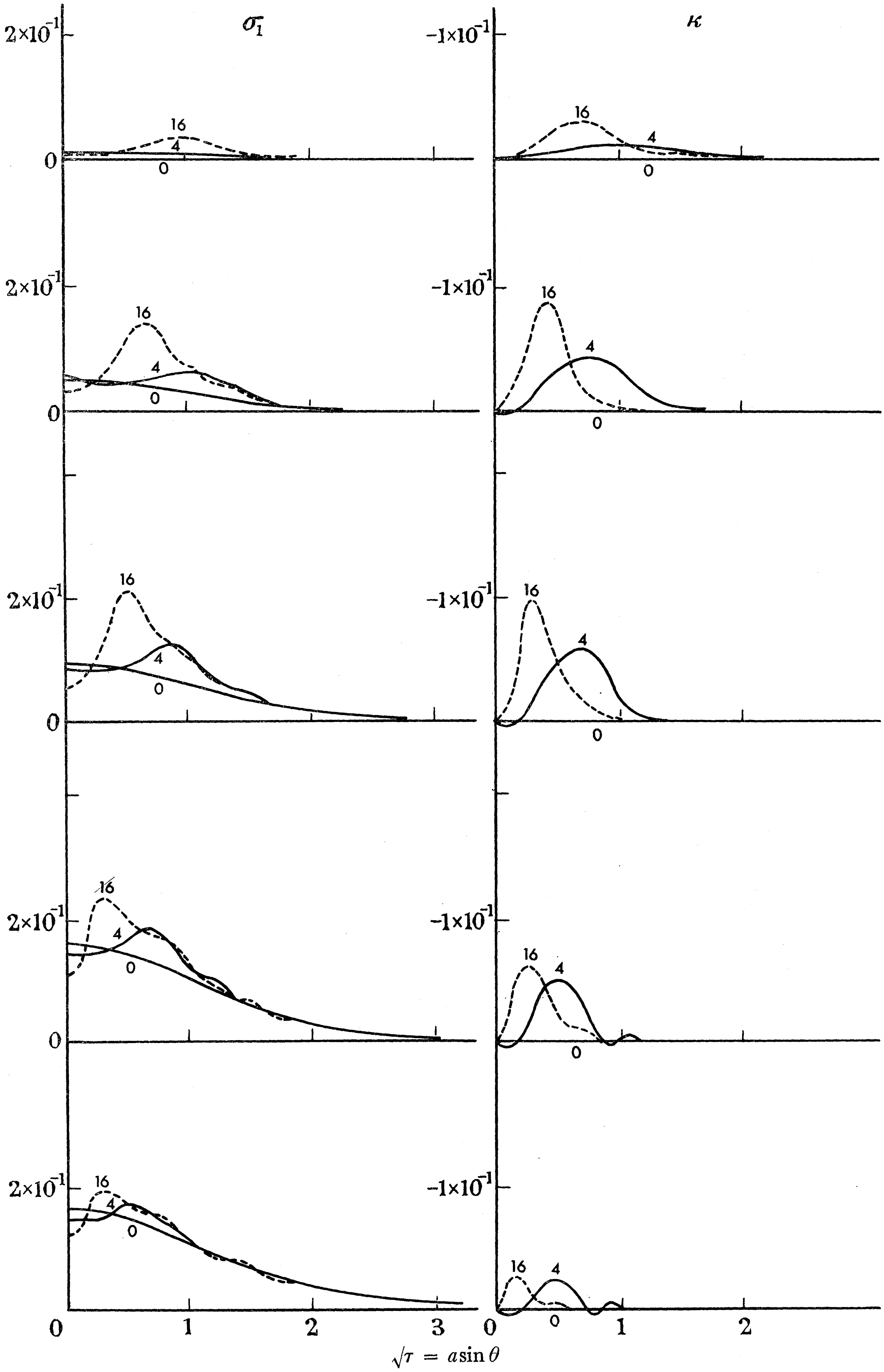

FIGURE 4 (*cont.*). For legend see facing page.

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The normalized probability distributions of amplitude $|E|$ and phase ψ of the wave field may now be found by setting the values of S_1, and S_2 and K obtained above into (2·3), integrating over all phases ψ to give $P(|E|)$ the amplitude probability distribution, and over all amplitudes $|E|$ to give $P(\psi)$ the phase distribution. This was done numerically and the distributions are given in figure 6 for different distances in the medium and for some

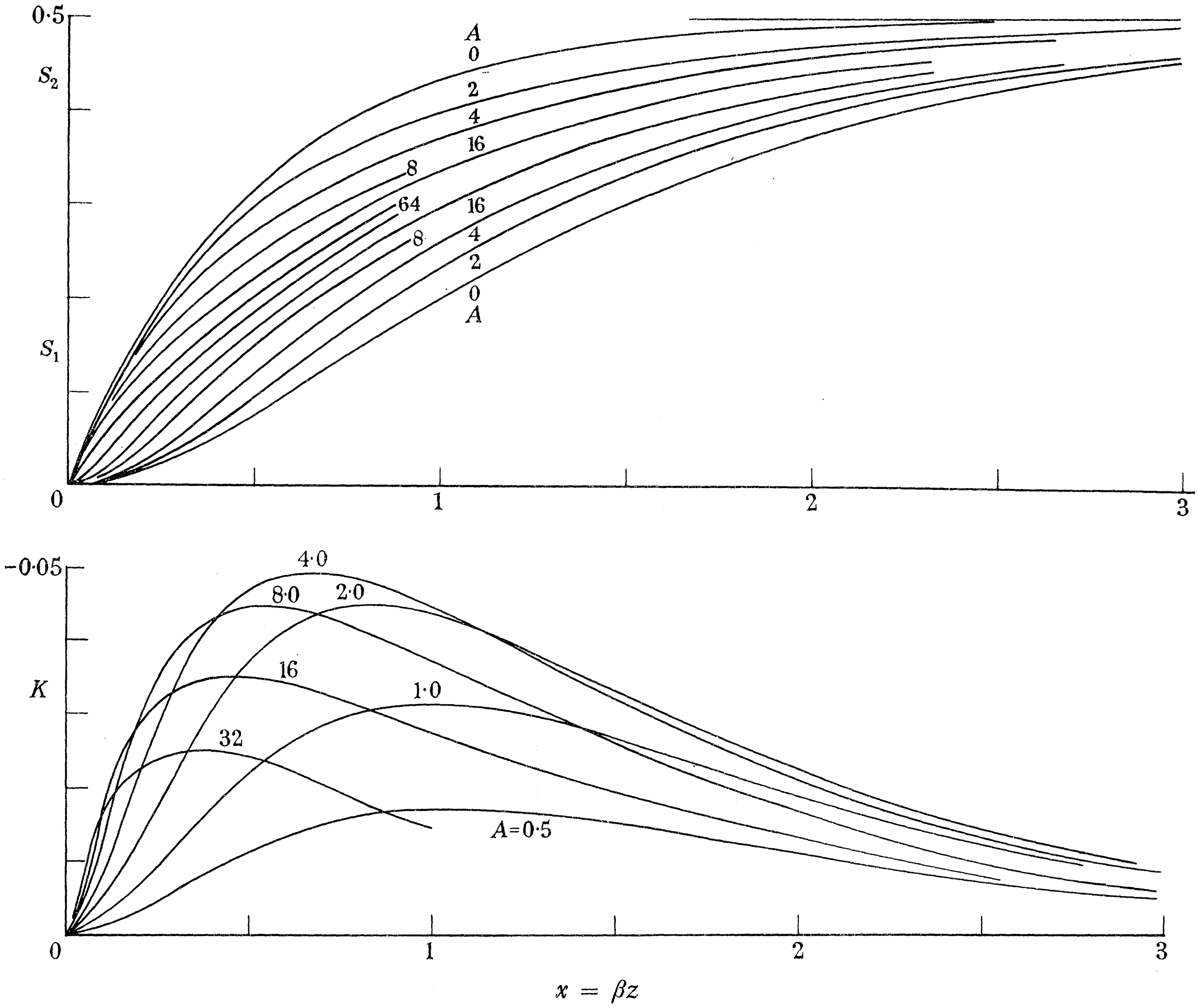

FIGURE 5. The functions S_1, S_2 and K for various values of A. Unity on the vertical scale denotes the value of $\Pi(0)$, the initial power flux of the incident wave. In the top figure the upper curves are those of S_2, the lower those of S_1.

representative values of the parameter A. As is to be expected the amplitude distribution approaches a Rayleigh distribution for large βz where the unscattered field becomes very small, while the phase distribution becomes uniform. For values of A and βz where K the correlation is not negligible, the phase distributions are asymmetrical. This is to be expected since for non-zero correlation K the equiprobability ellipse (see figure 1) is tilted with respect to the phase reference axes.

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Since the probability distributions are known, it is a simple matter to find any required average value of the amplitude or phase fluctuations at different depths in the medium. As an example the mean square fluctuation of amplitude defined by

$$\mathscr{E} = \frac{\langle |E(z)|^2\rangle - \langle |E(z)|\rangle^2}{\langle |E(z)|\rangle^2} \tag{10·1}$$

was calculated numerically for different values of A, using the probability distributions obtained above to get the necessary averages. The results are shown in figure 7.

For $\beta z \geqslant 2$, $\mathscr{E}$ approaches closely the limit 2·726 which is the value for a Rayleigh distribution of amplitude. Curves are also given in figure 8 for the mean square logarithm of the amplitude $\langle \ln^2 |E|\rangle$. This is of interest since the results of some authors are given in terms of this quantity (Chernov 1960; Tatarski 1961).

Average values of the mean square phase fluctuation $\langle \psi^2\rangle$ are given in figure 9.

It should be noted that there is comparatively small separation of the phase curves for different values of the parameter A, and this only in the region $\beta z \leqslant 2$. An expanded view of this region is given in figure 9. Thus the quantity $\langle \psi^2\rangle$ may be useful since it gives an estimate of the distance βz almost independently of the parameter A. Moreover it attains its limiting value of $\frac{1}{3}\pi^2$ (for uniform phase distribution) more slowly than any of the average quantities dependent on the amplitude, and so gives information on βz to greater depths in the medium.

The average phase fluctuation $\langle \psi\rangle$ does not seem to have been investigated previously, but is also of interest. Curves of this quantity are given in figure 10. It is interesting to note how closely they follow the corresponding curves for the correlation K. The average phase fluctuation $\langle \psi\rangle$ is non-zero due to the fact that the phase distribution $P(\psi)$ is asymmetrical when K is non-zero, and so the close connexion between $\langle \psi\rangle$ and K is not surprising.

It is thus a simple matter to find any average field quantity at a distance z in an inhomogeneous medium of the type treated above once β and A have been determined for the medium.

11. The axial ratio parameter D

The parameter $D = (r_z \lambda / 2\pi r_0^2)^2$ represents the effect of the axial ratio of the irregularities r_z/r_0 on the scattering. The results in this paper have been given for $D = 0$. The effect of non-zero D on these results, and in particular on the form of the angular spectrum of the scattered radiation will now be discussed.

The angular spectrum of the radiation scattered by an extended inhomogeneous medium of the type considered in this paper was derived by Fejer (1953). In the notation of the present treatment, the form he obtained was

$$\sigma_1 + \sigma_2 = \mathrm{e}^{-l} \sum_{j=1}^{\infty} \frac{l^j}{j!\,j} \exp\{-\tau/j\} \tag{11·1}$$

which is the solution of (7·9) with $D = 0$. Bramley (1954) showed that the same result could be obtained by 'collapsing' the medium treated by Fejer (1953) into a physically thin layer which imposed large phase modulations on the incident wave.

It was pointed out by Bowhill (1961 *b*), in §§ 2 and 3 of his paper, that both Fejer and Bramley had neglected the correlation of irregularities in the direction of wave propagation,

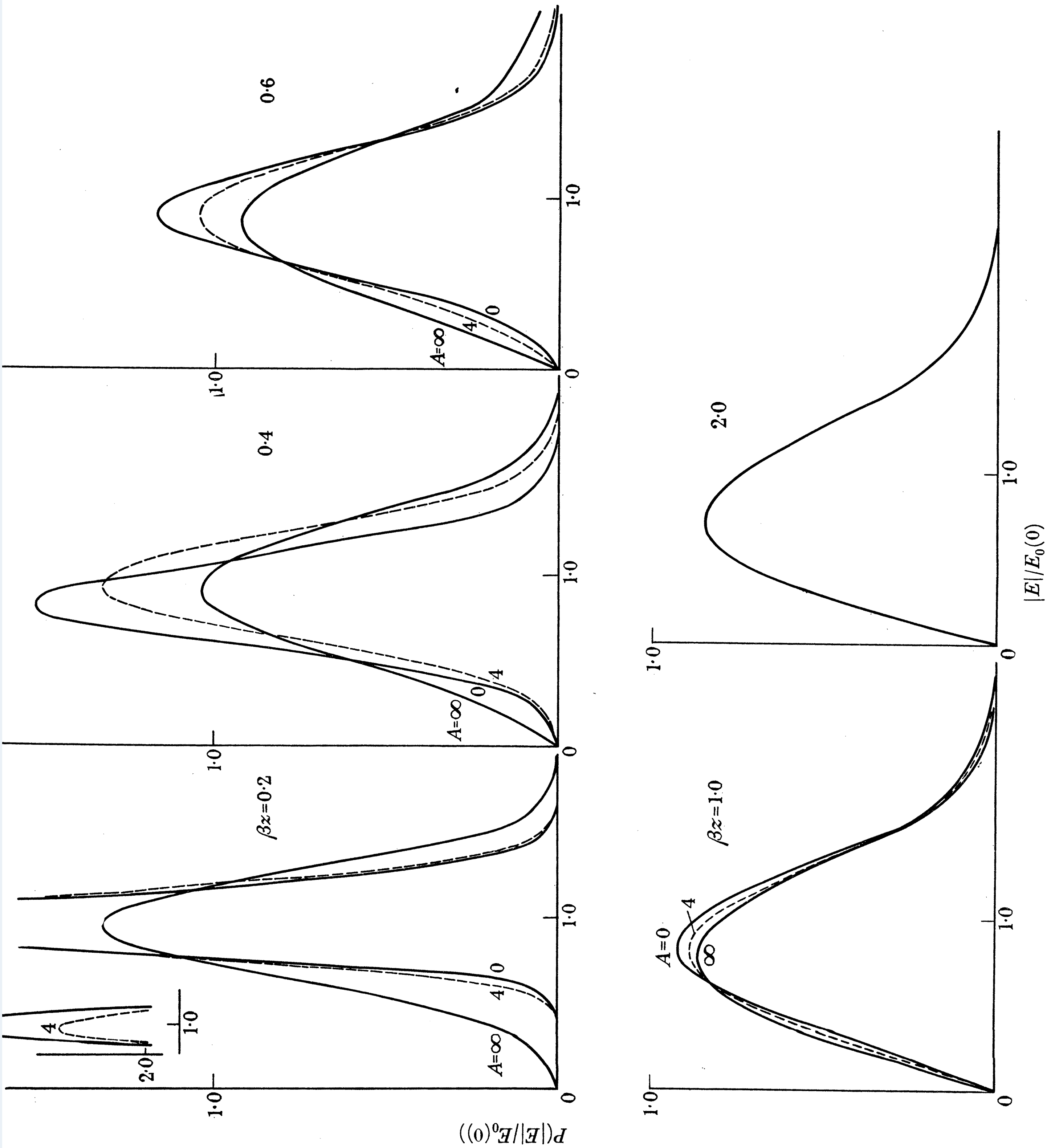

FIGURE 6. For legend see facing page

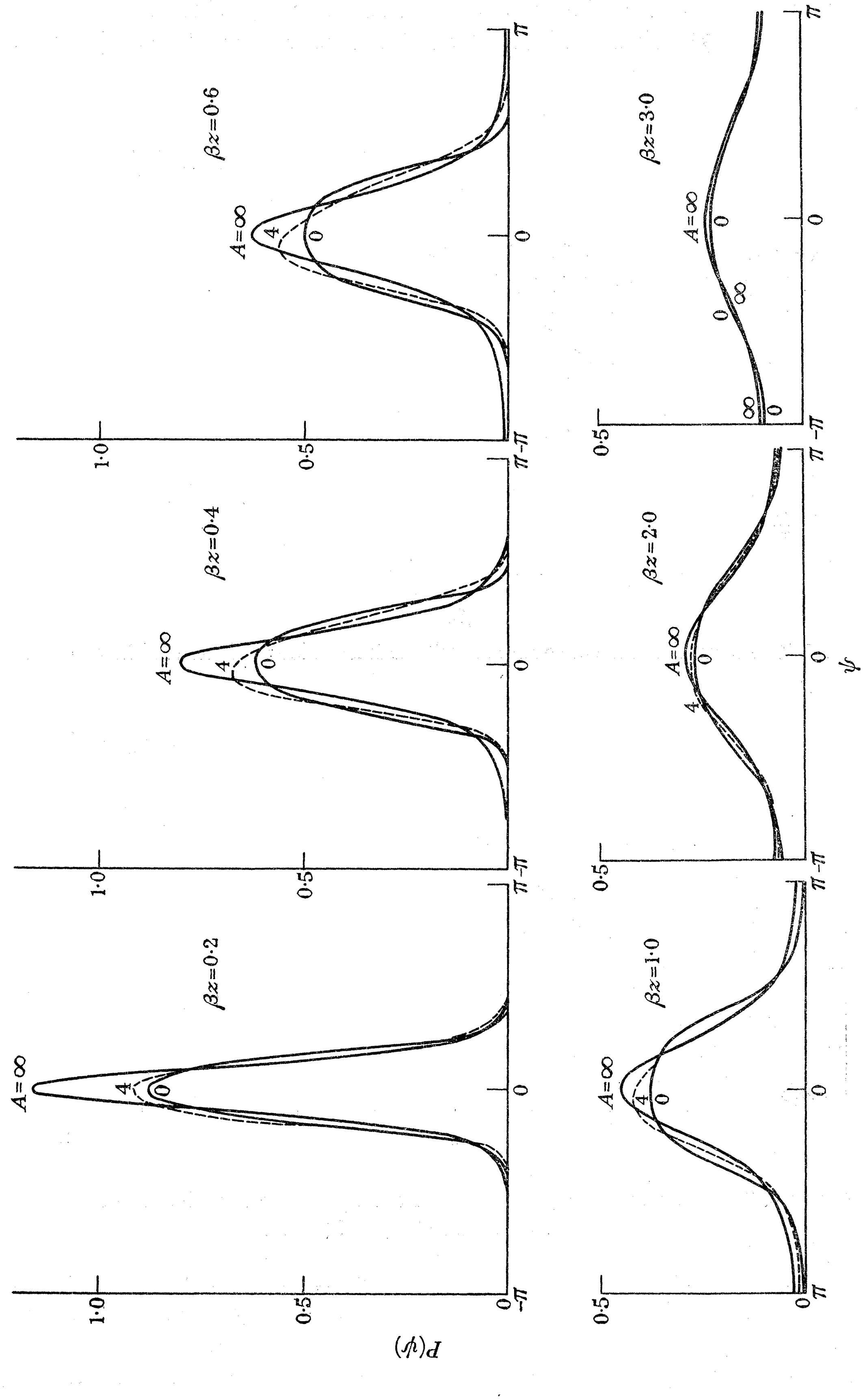

FIGURE 6 (*cont.*). The normalized probability distributions of amplitude $P(|E|/E_0(0))$ and phase $P(\psi)$ of the wave field at different distances in the medium for various values of A.

thus effectively assuming that $D = 0$. When correlation in the direction of wave propagation is allowed for it is no longer possible, as pointed out by Bowhill, to 'collapse' an extended medium into a thin layer. Taking this correlation into account Bowhill (1961 *b*) derived the

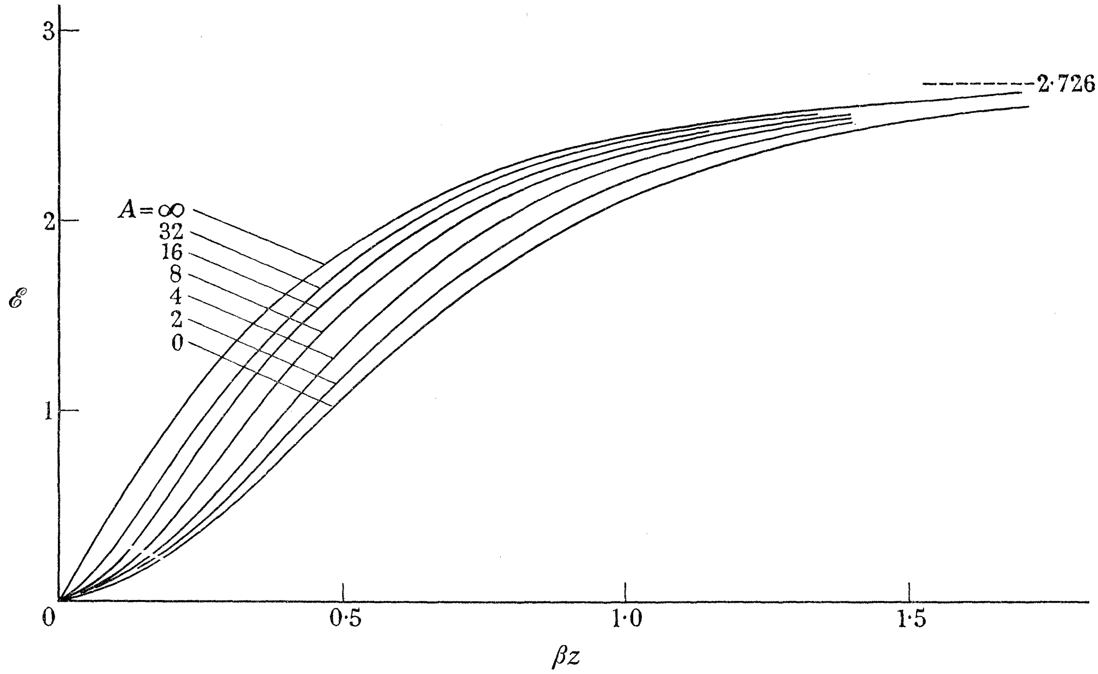

FIGURE 7. The normalized mean square amplitude fluctuation; $\mathscr{E}$ is defined in the text.

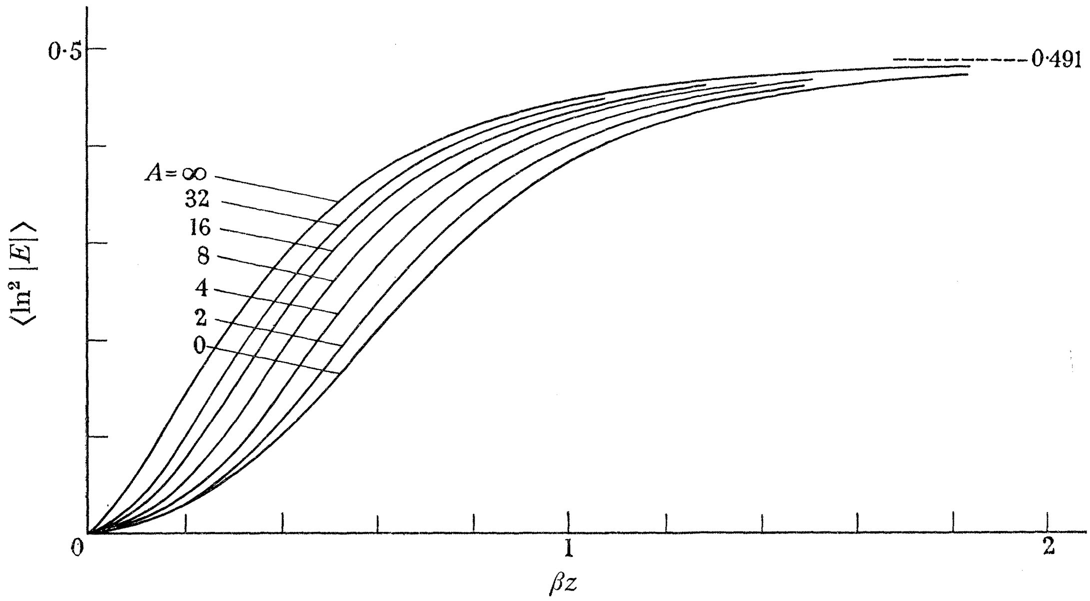

FIGURE 8. The mean square logarithm of the amplitude, the quantity calculated by Chernov (1960) and Tatarski (1961).

form of the angular spectrum for an extended layer containing very weakly scattering irregularities in the case when the incident radiation falls normally onto the layer. Section 4 derives the angular spectrum for such a layer in the case of oblique incidence and Bowhill's result may be obtained by setting $t' = 0$ in (4·2). The term in the exponent containing r_z

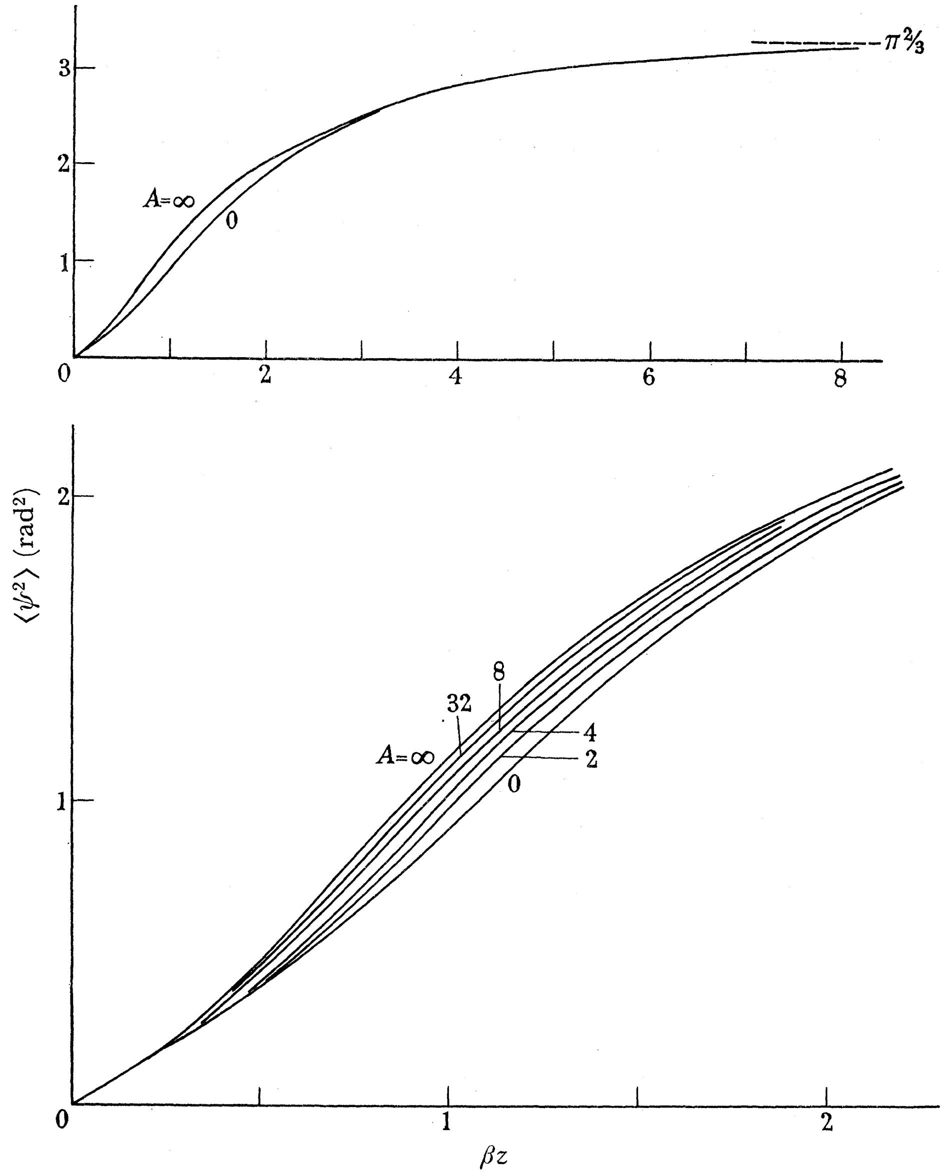

FIGURE 9. The mean-square phase fluctuation.

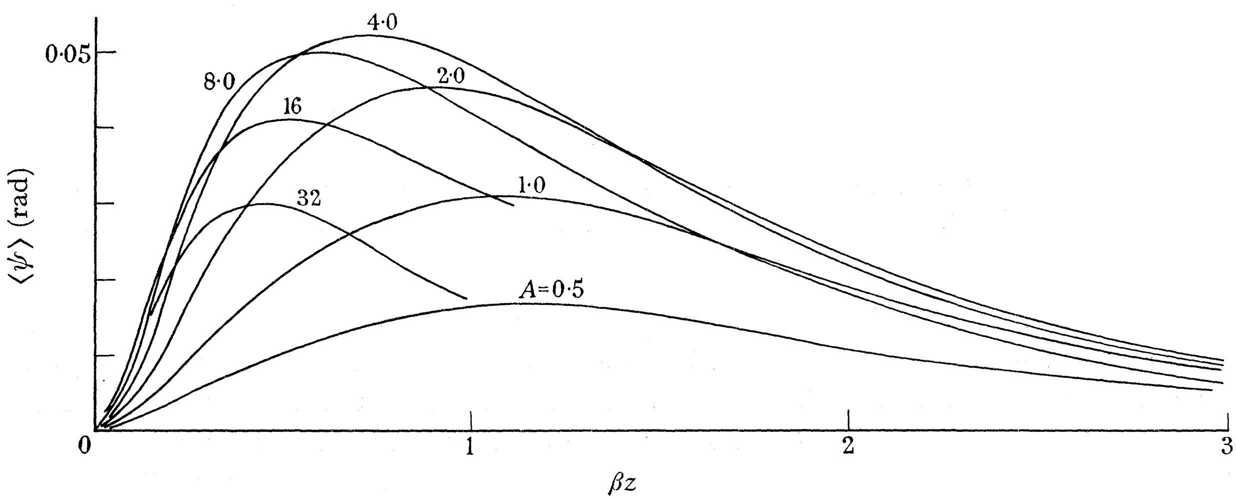

FIGURE 10. The mean phase fluctuation.

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gives the effect of correlation in the direction of wave propagation which, as pointed out by Bowhill (1961 *b*), leads to 'forward scatter' of the radiation and to a narrower angular spectrum.

The angular spectrum in Bowhill (1961 *b*), and that given by (4·2) refer to the case where single scatter only is of importance. The angular spectrum for multiply scattered radiation was not derived by Bowhill, but may be obtained by solving equation (7·9) with D non-zero.

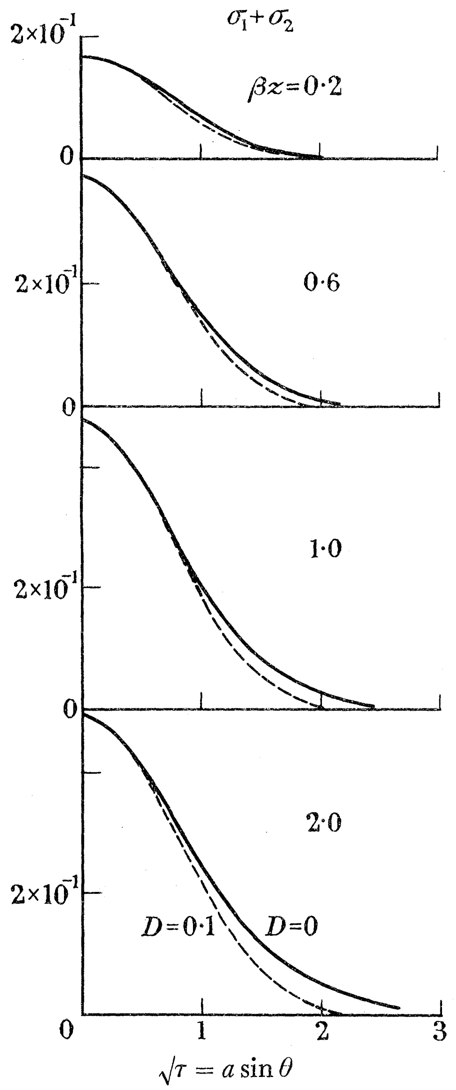

FIGURE 11. The angular spectrum for $D = 0$ and $D = 0\cdot1$. The horizontal scale is $a \sin\theta$, and unity on the vertical scale denotes the value of $\Pi(0)$, the initial power flux of the incident wave.

Since the smallest scale size of the irregularities must be much larger than the wave-length of the radiation, and since this paper deals with axial ratios of less than about 20, the maximum possible value of D is about 0·1. In figure 11 the angular spectrum is given for $D = 0$ and $D = 0\cdot1$ at different distances in the medium, and it is clear that while the angular spectrum is somewhat narrower when $D = 0\cdot1$ the effect is quite small over the region of interest, i.e. for $\beta z \leqslant 3$.

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The curves for S_1, S_2 and K have also been calculated for $D = 0{\cdot}1$ but differ only slightly from those for $D = 0$. This is the justification for neglecting D in §9 of the paper and presenting the results for the case $D = 0$. The results are only slightly different for $D = 0{\cdot}1$, and in a great many cases of interest D is very much smaller than this.

12. Conclusion

The present paper has shown that in a medium with weak random irregularities of refractive index the probability distributions of phase and amplitude of the wave field are functions of the quantities S_1, S_2 and K. A set of general integro-differential equations has been derived which enable S_1, S_2 and K to be found for a medium, provided that the autocorrelation function of the irregularities is known. The specific case of a medium having irregularities with a Gaussian autocorrelation function has been examined and S_1, S_2 and K found analytically for particular limiting cases of the parameter A. Values of S_1, S_2 and K for intermediate values of A have been obtained numerically and are presented, together with the corresponding probability distributions of the phase and amplitude fluctuations, and their average values at different distances in the medium.

The author is grateful to the University of Queensland for the award of a Travelling Scholarship; he is indebted to Dr K. G. Budden, F.R.S. of the Cavendish Laboratory for continued assistance and many helpful discussions, to Mr J. A. Ratcliffe, F.R.S. for valuable discussions, and to Dr A. Hewish of the Cavendish Laboratory, who pointed out the need for a theory of the type outlined in this paper.

He also wishes to thank Dr G. J. Daniell for advice on the numerical calculations, and Professor M. V. Wilkes, F.R.S. for permission to use the Titan Computer in the University Mathematical Laboratory, Cambridge.

Appendix A. The scattered field expressed as an angular spectrum of plane waves

The scattered field given by equation (3·8) will be expressed as an 'angular spectrum' of plane waves by taking its two-dimensional Fourier transform. The function $E_s(X, Y)$ in (3·8) is a statistically stationary, random function of X and Y, so that the usual Fourier integral expression gives a divergent integral. The function is therefore considered only within an area defined by $-\frac{1}{2}W < X < \frac{1}{2}W$, $-\frac{1}{2}W < Y < \frac{1}{2}W$ where W may be as large as desired. Outside this range the function $E_s(X, Y)$ may be imagined to repeat its behaviour within the range, so that it is doubly periodic, and may be represented by the double Fourier series

$$E_s(X, Y) = \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} \frac{F(k_x, k_y)}{W} \exp\{-\mathrm{i}(k_x X + k_y Y)\}, \tag{A 1}$$

where m, m' are integers and

$$k_x = 2\pi m/W, \quad k_y = 2\pi m'/W. \tag{A 2}$$

Then the coefficients F/W are obtained from the following expression for F:

$$F(k_x, k_y) = \frac{1}{W}\int_{-\frac{1}{2}W}^{\frac{1}{2}W}\int_{-\frac{1}{2}W}^{\frac{1}{2}W} E_s(X, Y) \exp\{\mathrm{i}(k_x X + k_y Y)\}\,\mathrm{d}X\,\mathrm{d}Y$$
$$= \frac{1}{W}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{s,Tr}(X, Y) \exp\{\mathrm{i}(k_x X + k_y Y)\}\,\mathrm{d}X\,\mathrm{d}Y, \tag{A 3}$$

where $E_{s,Tr}$ is the truncated value of E given by (3·8) for $-\frac{1}{2}W<X<\frac{1}{2}W$, $-\frac{1}{2}W<Y<\frac{1}{2}W$ and by zero outside this range (see, for example, Bendat (1958), p. 40). An expression for $E_{s,Tr}$ can be found by replacing the two pairs of limits $\pm\infty$ in (3·8) by $\pm\frac{1}{2}W$. This is permissible because W is to be made indefinitely large, and in particular $W \gg z$. This change of limits has the same effect as if the scattering layer were confined to the range

$$-\tfrac{1}{2}W < x_0 < \tfrac{1}{2}W, \quad -\tfrac{1}{2}W < y_0 < \tfrac{1}{2}W$$

so that there would be no scattered field if X, Y were outside this range. When later (equations (A 8) onwards) the squared modulus of (A 3) is used, the resulting factor $1/W^2$ on the right-hand side ensures that the expression is the power flux per unit area for small ranges dk_x, dk_y of k_x, k_y, and thus tends to a limiting value independent of W when $W\to\infty$.

If (3·8), with the limits modified as just indicated, is now inserted in (A 3), and if (3·7) is used:

$$F(k_x,k_y) = \frac{E_0k^2\exp(-ikz\sec\theta')}{2\pi W}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^L\int_{-\frac{1}{2}W}^{\frac{1}{2}W}\int_{-\frac{1}{2}W}^{\frac{1}{2}W}\frac{n_1(x_0,y_0,z_0)}{(z-z_0)\sec\theta'}\exp\{-ikH\}$$
$$\times\exp\{i[(k_x-k_x')X+(k_y-k_y')Y]\}\,dx_0\,dy_0\,dz_0\,dX\,dY. \quad \text{(A 4)}$$

The exponent of (A 4) is a quadratic function of X, Y, so that the X, Y integrations are a double complex Fresnel integral which can be evaluated by standard methods (see, for example, Budden 1965*b*, equation (11)) to give

$$F(k_x,k_y) = -\frac{iE_0\sec\theta'\exp(-ikz\sec\theta')}{W}k\int_0^L\int_{-\frac{1}{2}W}^{\frac{1}{2}W}\int_{-\frac{1}{2}W}^{\frac{1}{2}W}n_1(x_0,y_0,z_0)\exp\{i\alpha(z-z_0)\}$$
$$\times\exp\{i[(k_x-k_x')X_0+(k_y-k_y')Y_0]\}\,dx_0\,dy_0\,dz_0, \quad \text{(A 5)}$$

where $\alpha = [(k_x-k_x')^2+(k_y-k_y')^2+\tan^2\theta'\{(k_x-k_x')\cos\phi'+(k_y-k_y')\sin\phi'\}^2]\sec\theta'/2k. \quad$ (A 6)

The power flux in the z direction for this one Fourier component is

$$FF^*/2Z_0W^2,$$

where Z_0 is the characteristic impedance of free space. The number of component plane waves having k_x, k_y in the small ranges δk_x, δk_y, is $\delta m\,\delta m' = (W^2/4\pi^2)\,\delta k_x\,\delta k_y$ from (A 2). Thus the power flux for these components is

$$\frac{FF^*}{8\pi^2Z_0}\delta k_x\,\delta k_y, \quad \text{(A 7)}$$

where it is assumed that the angle between the wave normal and the z axis is small enough for the effects of obliquity and polarization to be ignored. The product FF^* is from (A 5),

$$FF^* = \frac{E_0^2\sec^2\theta'k^2}{W^2}\int(6)\int n_1n_1'\exp\{i[(k_x-k_x')(x_0-x_0')+(k_y-k_y')(y_0-y_0')-(\alpha+\gamma)(z_0-z_0')]\}$$
$$\times\,dx_0\,dy_0\,dz_0\,dx_0'\,dy_0'\,dz_0', \quad \text{(A 8)}$$

where $\gamma = [(k_x-k_x')\cos\phi'+(k_y-k_y')\sin\phi']\tan\theta'. \quad$ (A 9)

Appendix B. Integration when the autocorrelation function ρ is Gaussian

The integrals with respect to ξ, η in (3·19) converge rapidly so that the limits may be taken to be $\pm\infty$. The limits for X_1 and Y_1, however, remain $\pm\frac{1}{2}W$.

The ξ integration is

$$\int_{-\frac{1}{2}W}^{\frac{1}{2}W}\int_{-\infty}^{\infty}\exp\{-\xi^2/r_0^2\}\exp\{-\mathrm{i}\xi(k_x-k_x')\}\,\mathrm{d}\xi\,\mathrm{d}X_1 = Wr_0\sqrt{\pi}\exp\{-\tfrac{1}{4}r_0^2(k_x-k_x')^2\} \quad \text{(B 1)}$$

and the η integration is similar. The remaining integrals involving ζ, Z_1 give

$$\int_0^L\int_{-\infty}^{\infty}\exp\{-\zeta^2/r_z^2+\mathrm{i}(\alpha+\gamma)\,\zeta\}\,\mathrm{d}\zeta\,\mathrm{d}Z_1 = Lr_z\sqrt{\pi}\exp\{-\tfrac{1}{4}r_z^2(\alpha+\gamma)^2\}. \quad \text{(B 2)}$$

Use of the limits $\pm\infty$ for ζ (Obukhov 1953) means that $L \gg r_z$. Thus the thickness of the elementary slab must be chosen to be much larger than the correlation distance of the irregularities in the direction of wave propagation since powers scattered from near the front and near the back of the layer are then mutually incoherent.

The integrals (B 1) and (B 2) are set in (3·19) and (3·6) is employed to give the result (3·20) in § 3 of the paper.

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